



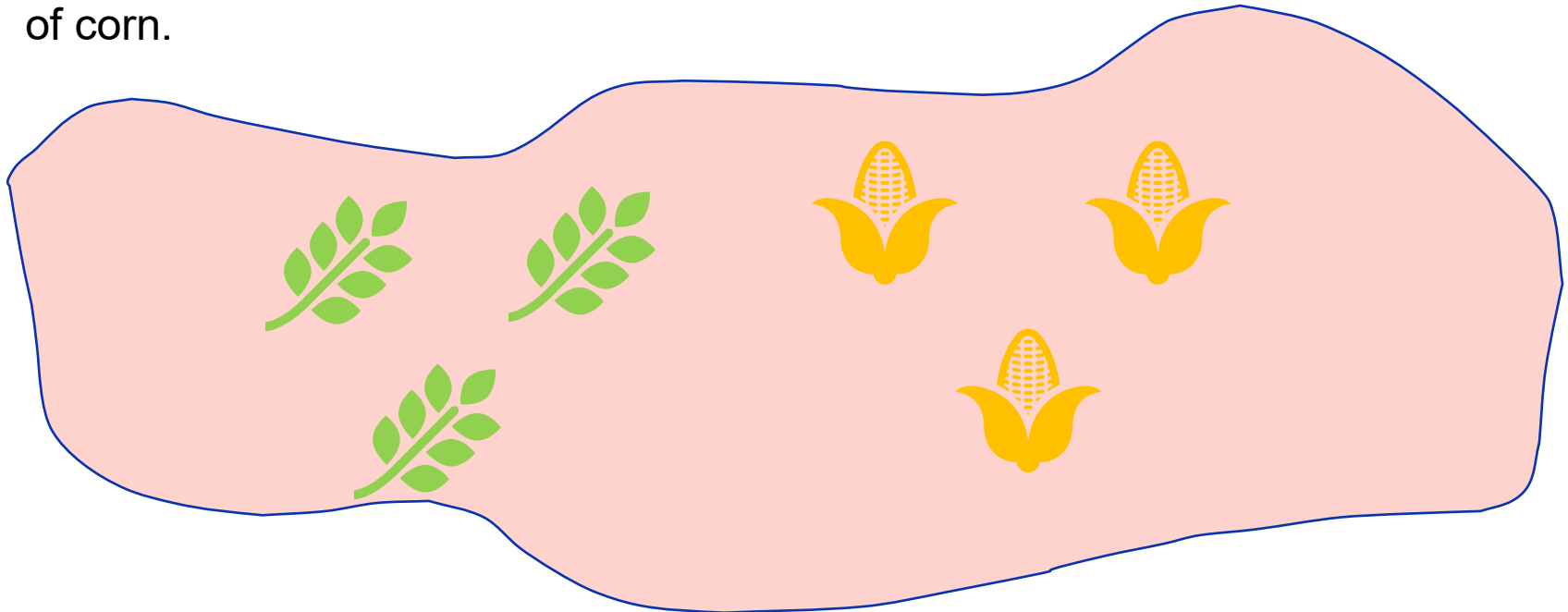
Introduction to Optimisation

Lecture 1

Lecture notes by Dr. Julia Memar and Dr. Hanyu Gu and with an acknowledgement to Dr.FJ Hwang and Dr.Van Ha Do

Let's start with an example...

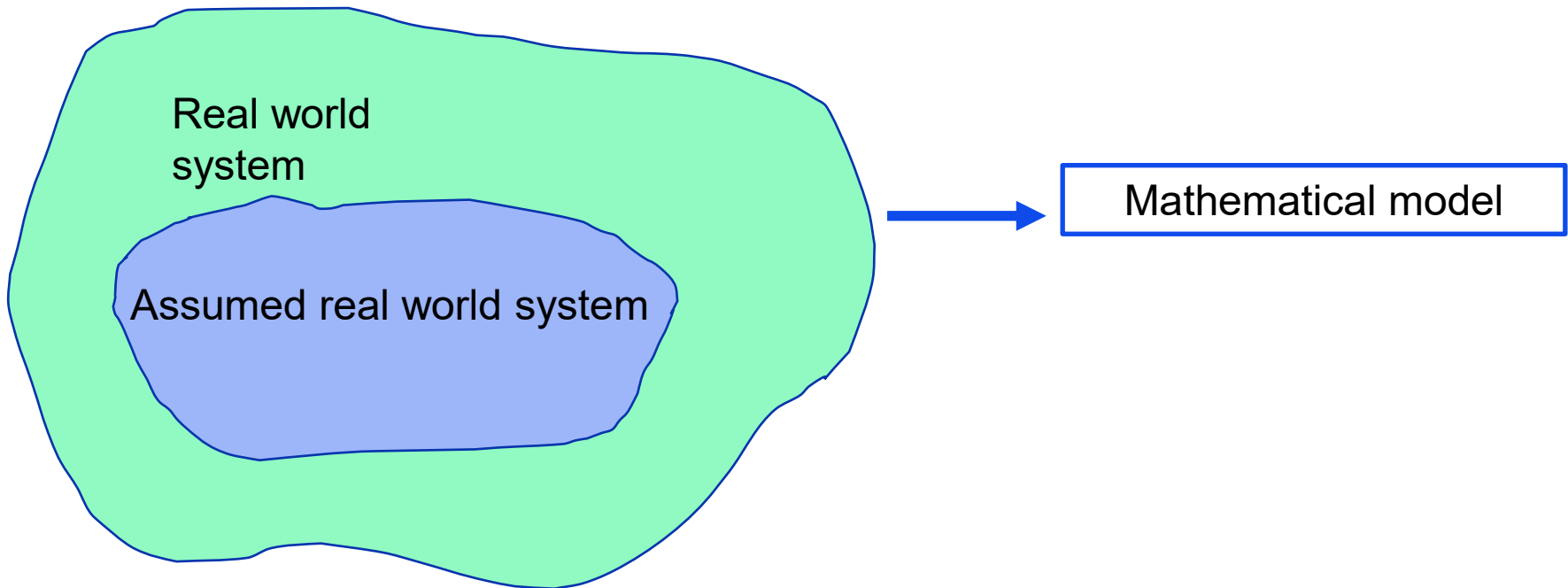
Farmer Jones plants wheat and corn 4 acres of land. The selling prices are \$8 and \$5 per tonne of wheat and corn, respectively. The yields per acre are 2 tonnes and 3 tonnes for wheat and corn, respectively. The farm cannot sell more than 4 tonnes of wheat and more than 9 tonnes of corn.



What is the maximal revenue that the farm can achieve?

What is Mathematical Optimisation?

- Mathematical optimisation (or mathematical programming) is a field of applied mathematics that is concerned with solution of quantitative problems.



What is Mathematical Optimisation?

- The general form of the mathematical programming problems is the following:

$$\max(\text{or } \min) \quad z = f(x)$$

subject to $g(x) \leq (\text{or } \geq, \text{or } =) 0,$

$x \geq 0, (\text{or } \leq 0, \text{or } \text{urs} - \text{unrestricted in sign})$

where x and $g(x)$ are vectors.

What is Mathematical Optimisation?

Example:

$$\max z = f(x_1, x_2, x_3, x_4) \quad \leftarrow \text{Objective function}$$

s.t.

$$\left. \begin{aligned} g_1(x_1, x_2, x_3, x_4) &\leq 0, \\ g_2(x_1, x_2, x_3, x_4) &\leq 0, \\ g_3(x_1, x_2, x_3, x_4) &\leq 0, \\ g_4(x_1, x_2, x_3, x_4) &\leq 0, \end{aligned} \right\} \leftarrow \text{Constraints}$$

The set $S = \{x : g(x) \leq 0 \text{ and } x \geq 0\}$ is called ***the feasible region***.

Observe that

$$z = \max_{x \in S} f(x) = -\min_{x \in S} (-f(x))$$

What is Mathematical Optimisation?

Examples:

$$\begin{array}{l} \text{max} \quad 2x_1 + 3x_2 \\ \text{s.t.} \quad x_1 + 3x_2 \leq 6 \\ \quad \quad 3x_1 + 5x_2 \leq 15 \\ \quad \quad x_1, x_2 \geq 0 \end{array}$$

$$\begin{array}{l} \text{min} \quad x_1^3 + 2x_1^2x_2 + 5x_2^4 \\ \text{s.t.} \quad 2x_1 + x_2 \geq 4 \\ \quad \quad 3x_1^2 + 2x_2 \geq 5 \\ \quad \quad x_1, x_2 \geq 0 \end{array}$$

$$\begin{array}{l} \text{max} \quad 2x_1 - 2x_1x_2 + 5x_2 \\ \text{s.t.} \quad x_1 + x_2 \leq 4 \\ \quad \quad x_1, x_2 \geq 0, x_1, x_2 \text{ integer} \end{array}$$

How to solve an optimization problem?

Formulate the problem:

- Define decision variables
- Express objective function and constraints using the decision variables
- Solve the problem (easy to say....)
 - Graphical method
 - Simplex
 - Excel (Solver)
 - Lingo
 - CPLEX or other optimization software
 - ...and more

Applications

- Blending petroleum products: maximise the profit subject to constraints on quantity of components of petroleum available (LP).
- Portfolio design: maximise the expected return subject to constraints on maximum levels of risk acceptable (NLP).
- Aircrew scheduling: minimise the cost to an airline (wages plus accommodation expenses, etc) subject to all flights having the required crew, all crew returning to their home bases, and all union and legal requirements on work schedules met (IP).
- What do you think?

Complications

Consider the problem:

$$\max f(x) = x^3 - 3x$$

$$\text{s.t.} \quad x \leq 2$$

$$x \geq 0$$

The feasible region is $[0, 2]$, the end points are $x = 0$ and $x = 2$

$f'(x) = 3x^2 - 3$; the stationary points are $x = -1$ and $x = 1$

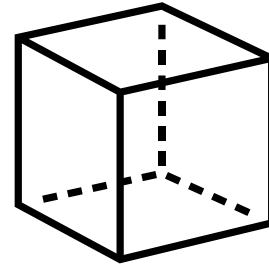
$$\max f(x) = \max\{f(0), f(1), f(2)\}$$

Complications

- The Abel–Ruffini theorem states that there is no solution in radicals to general polynomial equations of degree five or higher with arbitrary coefficients.
- Much more challenging in high dimensions (large number of variables)

Complications

- How many extreme points in a square?
- How many extreme points in a cube?
- How many extreme points in n – *dimensional hypercube*?
- How to find a solution without full enumeration of extreme points?



Blending problem

A one-kilogram pack of dog food must contain protein (at least 31%), carbohydrate (at least 38%), and fat (between 13% and 15%, that is greater than or equal to 13% and less than or equal to 15%). Three foods (food1, food2 and food3) are to be blended together in various proportions to produce a least-cost pack of dog food satisfying these requirements. The information for one kilogram of each food is given in the table below.

	<i>protein</i> (grams)	<i>carbohydrate</i> (grams)	<i>fat</i> (grams)	<i>price</i> (dollars)
food1	300	500	90	3
food2	450	300	160	1
food3	200	400	200	2

Blending problem 2

A company should produce steel from the following three alloys:

	cost (per tonne)	carbon	chrome	silicon	available (tonne)
Alloy 1	\$52	5%	0.1%	3%	unlimited
Alloy 2	\$71	3.8%	0.4%	2%	2000
Alloy 3	\$46	2%	0.3%	2.8%	5000

The chemical content of a blend is the weighted average of the chemical content of its components. For example, the portion of carbon in a blend of 20 tonnes of Alloy 1 and 30 tonnes of Alloy 2 is

$$\frac{20 \times 0.05 + 30 \times 0.038}{20 + 30} = 0.0428, \quad \text{that is } 4.28\%.$$

The steel should satisfy the following quality requirements

and will be sold for \$65 per tonne.

The company wishes to maximise its profit.

	at least	not more than
carbon	3.5%	4 %
chrome	0.2%	0.5 %
silicon	2.7%	2.7 %

Production planning

Giapetto's wooden soldiers and trains. Each soldier sells for \$27, uses \$10 of raw materials and takes \$14 of labor & overhead costs. Each train sells for \$21, uses \$9 of raw materials, and takes \$10 of overhead costs. Each soldier needs 2 hours finishing and 1 hour carpentry; each train needs 1 hour finishing and 1 hour carpentry. Raw materials are unlimited, but only 100 hours of finishing and 80 hours of carpentry are available each week. Demand for trains is unlimited; but at most 40 soldiers can be sold each week. How many of each toy should be made each week to maximize profits?

(Winston 3.1, p. 49)



Integer programming modelling: Portfolio selection problem

Over a 3-year planning horizon, an investor can invest in the following projects:

project	investment			return
	Year 1	Year 2	Year 3	
Project 1	2	8	4	35
Project 2	5	4	10	22
Project 3	9	3	13	39
Project 4	1	5	7	30
Project 5	11	6	3	31
Project 6	4	4	6	19
Project 7	2	12	4	20



The following funds are available over this 3-year planning horizon:

	Year 1	Year 2	Year 3
available funds	33	40	41

Integer programming modelling: workforce planning

A company has only full-time employees. The following table specifies the number of personnel required each day:

	required number
Monday	20
Tuesday	13
Wednesday	10
Thursday	12
Friday	15
Saturday	9
Sunday	17



If each employee works five days per week, what is the minimum number of employees needed in order to meet the daily requirements?

Portfolio selection problem

The investment strategy should satisfy the following requirements:

- ▶ the investor can invest in at most four projects;
- ▶ if the investor invests in Project 1, then he can invest in at most two of the five projects - Project 2, Project 3, Project 4, Project 5 and Project 7;
- ▶ if the investor invests in Project 2, then he must also invest in at least two of the three projects - Project 3, Project 4 and Project 6;
- ▶ if the investor invests in either Project 5 or Project 6 or in both, then he must invest in at least one of the two projects - Project 2 or Project 4.
- ▶ if the investor invests in Project 1, then he cannot invest in Project 6;
- ▶ if the investor invests in one of the two projects - Project 1 or Project 3, then he must also invest in the other one;

The investor wishes to maximise the total return from his investments.



Modeling: a summary

The steps involved in the formulation of a mathematical optimisation model are:

- a. Identify the decision variables (whose quantities can be controlled by the decision maker).
- b. Construct the objective function and constraints (restrictions) in terms of the decision variables. (You should not introduce new variables (unknowns) at this point.
- c. Ask yourself whether there is any hidden assumption that may exist, e.g. integer decision variables, fractional decision variables.
The most common “hidden” assumption is that the variables are nonnegative.
- d. Write the complete formulation as a linear program, objective first, followed underneath by the constraints.
- e. It is usually a good idea to write the formulation out first before constructing the spreadsheet until you get experience at formulation and structuring spreadsheets.