Introduction to Optimisation:

Constrained Nonlinear Programming

Lecture 9

Lecture notes by Dr. Julia Memar and Dr. Hanyu Gu and with an acknowledgement to Dr.FJ Hwang and Dr.Van Ha Do

Introduction

- Let $f(x)$ is be nonlinear function of vector $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ defined over the domain $D \subseteq R^n$. Consider an NLP problem

$$\boxed{\begin{aligned} &\min z = f(x) \\ &\text{s.t. } Ax = b, \end{aligned}}$$

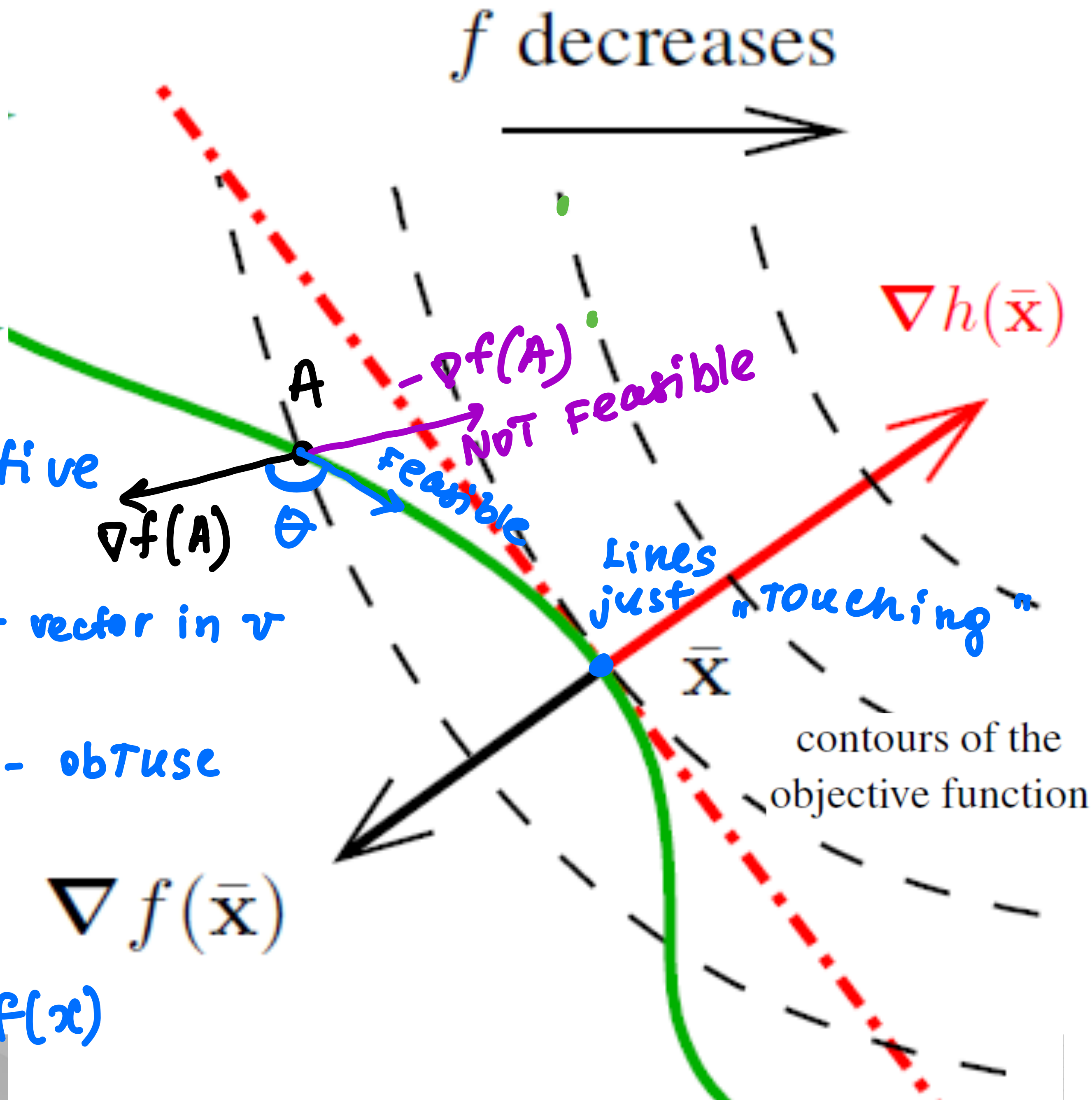
where A is $m \times n$ matrix, $rank\ A = m$

- Assume that $f(x)$ has continuous second-order partial derivatives for each point $x = (x_1, x_2, \dots, x_n)$ in $D = \{x : Ax = b\}$

Introduction

Improving direction and
Feasible direction:

$S = \{x : h(x) = 0\}$
constraints



Directional Derivative

$D_v(x) = \nabla f(x) \cdot \underset{\sim}{v} =$

↑ unit vector in v

$= |\nabla f(x)| \cos\theta$

if $90^\circ < \theta < 180^\circ$ - obtuse

then $\cos\theta < 0$

↓

v is direction of decrease of f(x)

Preliminaries

- Null space of $A_{m\times n}$, $n \geq m$ is

$$N(A) = \{p: Ap = 0\}$$

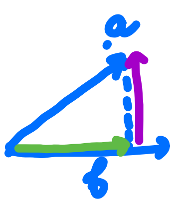
- Range space of $A_{m\times n}$

$$R(A) = \{q \in \Re^n: q = \underline{A^T\Lambda}, \Lambda \in \Re^m\}$$

- $N(A)$ and $R(A^T)$ are orthogonal subspaces: for $q \in R(A)$ and $p \in N(A)$:

$$q^T p = \Lambda A p = 0$$

- Any $x \in \Re^n$: $x = p + q$

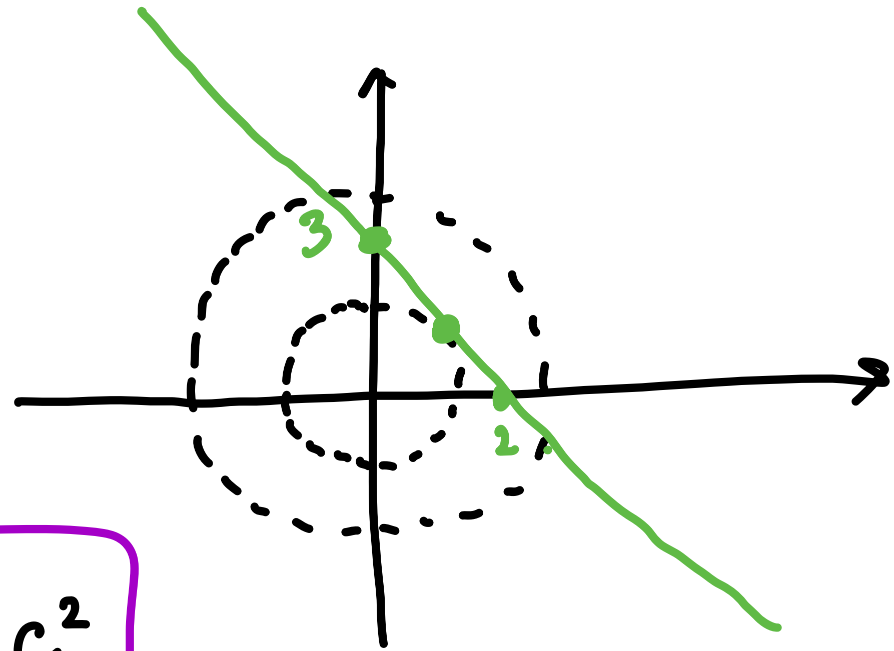


$a = \text{proj}_b a + a_\perp$

Reduced function

Example:

➢ $\left.\begin{aligned} &\min f(x_1, x_2) = x_1^2 + x_2^2 \\ &\text{s.t. } 3x_1 + 2x_2 = 6 \end{aligned}\right\} (*)$



$$f(x_1, x_2) = \boxed{x_1^2 + x_2^2 = c^2}$$

contour lines are circles

$3x_1 + 2x_2 = 6 \rightarrow$

- $x_2 = 3 - \frac{3}{2} x_1$

$$\Phi(x_1) = x_1^2 + \left(3 - \tfrac{3}{2}x_1\right)^2 \rightarrow \text{find } \min \Phi(x_1)$$

Find where $\Phi' = 0$

$$\Phi'(x_1) = 2x_1 + 2\left(3 - \tfrac{3}{2}x_1\right) \times \left(-\tfrac{3}{2}\right) = 0.$$

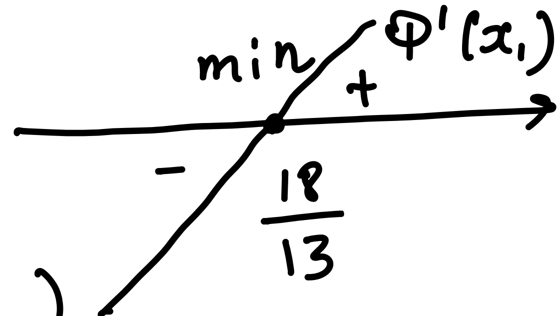
$$2x_1 - 9 + \frac{9}{2}x_1 = 0. \qquad \times 2$$

$$4x_1 - 18 + 9x_1 = 0$$

$$13x_1 = 18$$

$$x_1 = \frac{18}{13} \rightarrow x_2 = 3 - \frac{3}{2} \times \frac{18}{13} = \frac{12}{13}$$

$x_1 = \frac{18}{13}$ is min as



minimiser of $f(x_1, x_2)$ on Domain is

$$x^* = \begin{pmatrix} 18/13 \\ 12/13 \end{pmatrix}$$

Reduced function

In general form: $Ax = b$

- Choose x_B , then $x = \begin{pmatrix} x_B \\ x_N \end{pmatrix} \to Ax = b \to (B \mid N)\begin{pmatrix} x_B \\ x_N \end{pmatrix} = Bx_B + Nx_N = b$

$$\downarrow$$

$$x_B = B^{-1}b - B^{-1}Nx_N$$

$$x = \begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} B^{-1}b - B^{-1}Nx_N \\ x_N \end{pmatrix} = \underbrace{\begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix}}_{\bar{x}} + \underbrace{\begin{pmatrix} -B^{-1}N \\ I \end{pmatrix}}_{Z} x_N .$$

- $x = \bar{x} + Zx_N$ (particular and homogeneous solutions)

$$Ax = b$$

$$A(\bar{x} + Zx_N) = \underbrace{A\bar{x}}_{= b} + \underbrace{A(Zx_N)}_{= 0} = b .$$

Reduced function

also reduced The number of variables

- Reduced cost function is

$$\Phi(x_N) = f(\bar{x} + Z x_N)$$

instead of solving $\underbrace{\min f(x) \text{ s.t. } Ax = b}_{\text{constrained}}$ solve $\underbrace{\min \Phi(x_N)}_{\text{unconstrained}}$

- The matrix Z is basis matrix of the null space of $A_{m\times n}$

as Zx generates all $p = Zx : Ap = 0$

Null-space basis matrix for $A_{m\times n}$ is the $n \times (n-m)$ matrix Z: $\begin{pmatrix} -B^{-1}N \\ I \end{pmatrix}$

some ↓ particular s-n

- In other words, if $A\bar{x} = b$, then any feasible point $x = \bar{x} + p$, where $p \in N(A)$

- Hence Zx_n and $-Zx_n$ are all possible feasible directions for an arbitrary x_n.

Reduced function

Example:

- $\min f(x_1, x_2, x_3) = x_1^2 + 4x_1x_3 + x_2^2$

$$\text{s.t.} \quad \begin{aligned} 2x_1 + x_2 + 4x_3 &= 5 \\ 3x_1 + x_2 - x_3 &= 1 \end{aligned} \quad (*)$$

- The feasible set is all x satisfying $(*)$

$$A = \begin{pmatrix} 2 & 1 & 4 \\ 3 & 1 & -1 \end{pmatrix} \quad b = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

we are going to constructed reduced function, and hence solve the problem:

$$|x_B| = 2$$

$$x_B = (x_1, x_2) \; ; \quad x_N = x_3$$

$$B = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} ; \quad B^{-1} = -\begin{pmatrix} 1 & -1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix}$$

$$N = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

$$B^{-1}N = \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} -5 \\ 14 \end{pmatrix} ;$$

$$B^{-1}b = \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 13 \end{pmatrix}$$

$$x = \underbrace{\begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix}}_{\text{particular s-n of } Ax=b} + \underbrace{\begin{pmatrix} -B^{-1}N \\ I \end{pmatrix}}_{\text{general s-n of } Av=0} x_3 =$$

$$= \begin{pmatrix} -4 \\ 13 \\ 0 \end{pmatrix} + \underbrace{\begin{pmatrix} 5 \\ -14 \\ 1 \end{pmatrix}}_{z} x_3 \rightarrow \begin{aligned} x_1 &= -4 + 5x_3 \\ x_2 &= 13 - 14x_3 \end{aligned} \quad (*)$$

$\downarrow$ sub-IN to $f(x_1, x_2, x_3)$

$$\Phi(x_3) = (-4+5x_3)^2 + 4(-4+5x_3)\,x_3 + (13-14x_3)^2$$

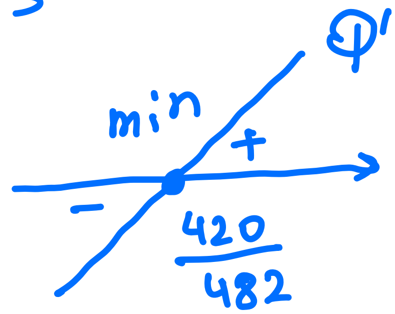
Find $\min \Phi(x_3) \rightarrow \Phi' = 0$

$$\Phi'(x_3) = 2(-4+5x_3)\times 5 - 16 + 40x_3 + 2(13-14x_3)\times(-14) =$$

$$= -40 + 50x_3 - 16 + 40x_3 - 364 + 392x_3 =$$

$$= -420 + 482x_3$$

$\Phi' = 0$ when $x_3 = \dfrac{420}{482}$



Hence minimiser of $f(x_1\ x_2\ x_3)$ is

$$x = \begin{pmatrix} -4 + 5\times\dfrac{420}{482} \\ 13 - 14\times\dfrac{420}{482} \\ \dfrac{420}{482} \end{pmatrix}$$

Reduced function

Example:

- $\min f(x_1, x_2, x_3) = x_1^2 + 4x_1x_3 + x_2^2$

$$\text{s.t.} \qquad 2x_1 + x_2 + 4x_3 = 5$$
$$3x_1 + x_2 - x_3 = 1$$

$$(A \mid b) \sim (I \mid A^{-1}b)$$

- OR use Gaussan- Jordan reduction

$$\left(\begin{array}{ccc|c} 2 & 1 & 4 & 5 \\ 3 & 1 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1/2 & 2 & 5/2 \\ 0 & -1/2 & -7 & -\frac{13}{2} \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1/2 & 2 & \frac{5}{2} \\ 0 & 1 & 14 & 13 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -5 & -4 \\ 0 & 1 & 14 & 13 \end{array}\right) \rightarrow \begin{cases} x_1 - 5x_3 = -4 \\ x_2 + 14x_3 = 13 \end{cases}$$

same as above (*)

Optimality conditions

by Th. 6, if in x^* $\nabla f = 0$ and $\nabla^2 f$ is pos-def $\rightarrow$ x^* is min

- Unconstrained NLP problem with a reduced function:

$$\min \phi(x_N)$$

where $\phi(x_N) = f(\bar{x} + Z x_N)$

- To set optimality conditions find
  1. *Reduced gradient* $\nabla \phi(x_N) =$ $Z^T \nabla f(x_N)$
     gradient of reduced f-n
  2. *Reduced Hessian* $\nabla^2\phi(x_N) =$ $Z^T \nabla^2 f(x_N) Z$
     Hessian of reduced f-n

UTS

Theorem 1 - constrained

Optimality conditions

Theorem 1. (Second-order necessary conditions – Linear equality constraints)

➢ If x^* a local minimiser of $f(x)$ over the set $\{x : Ax = b\}$, and Z is a basis matrix for the null-space of A, then

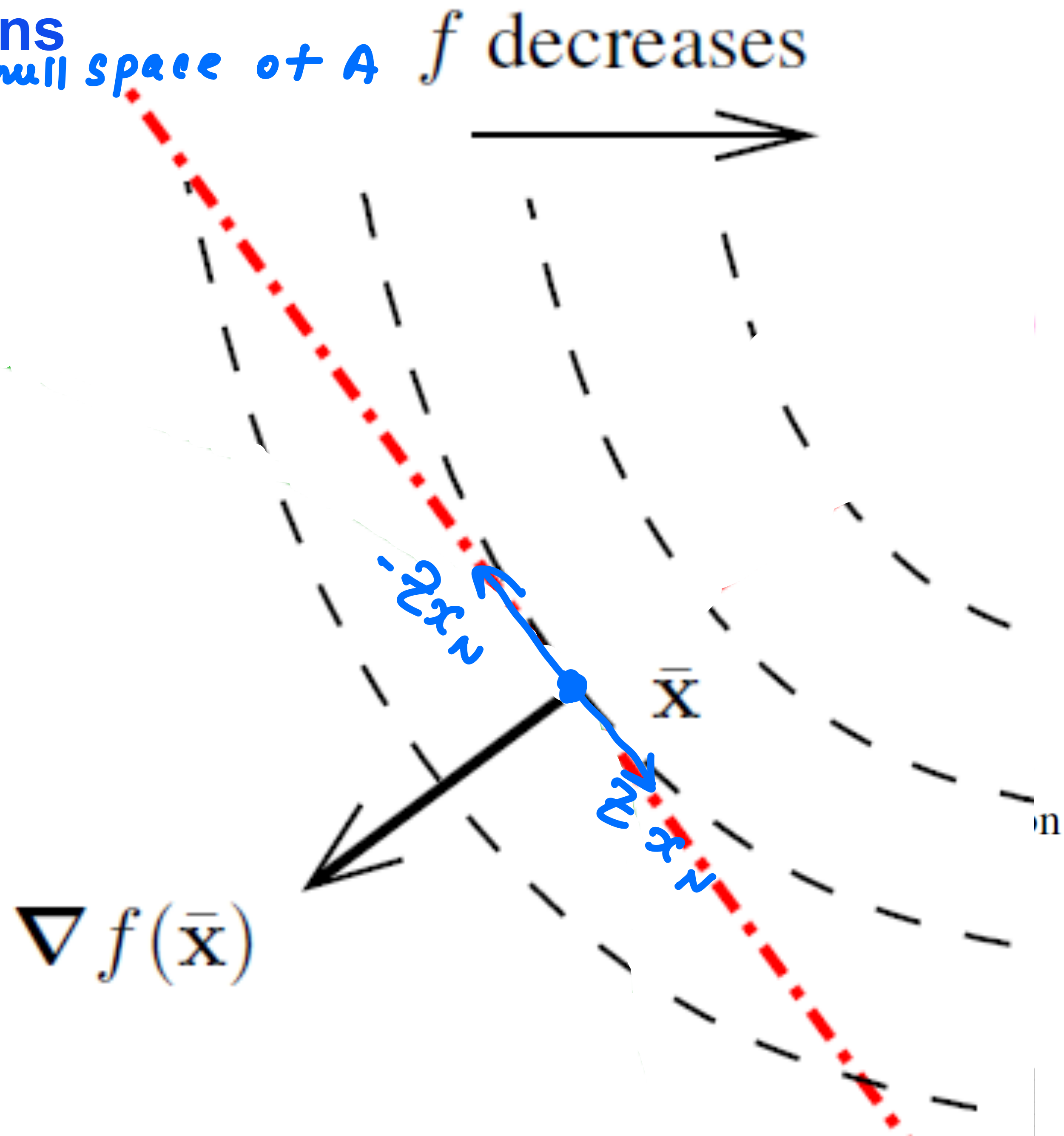
i. $Z^T(\nabla f(x^*)) = 0$, and

ii. $Z^T\nabla^2 f(x^*)Z$ is positive semidefinite.

Optimality conditions

orthogonal to null space of A

$\nabla f(\bar{x})$ is __________ to $N(A)$



Theorem 2 - constrained

Optimality conditions

Theorem 2. (Second-order sufficient conditions – Linear equality constraints)

- If Z is a basis matrix for the null-space of A and the point x^* satisfies

  i. $Ax^* = b$ → x^* is feasible

  ii. $Z^T \nabla f(x^*) = 0$, and

  iii. $Z^T \nabla^2 f(x^*) Z$ is positive-definite.

then x^* a local minimiser of $f(x)$ over the set $\{x : Ax = b\}$.

Observe that given a point x for a considered linear-equality constrained NLP problem we can apply directly the above two theorems without deriving a reduced function.

If $B = 2$; $B^{-1} = \frac{1}{2}$

Optimality conditions - example

- $\min f(x_1, x_2, x_3) = x_1^2 - 2x_1 + x_2^2 - x_3^2 + 4x_3$

  s.t. $\quad x_1 - x_2 + 2x_3 = 2$ $\qquad A = (1 \quad -1 \quad 2)$

- $\nabla f(x) = \begin{pmatrix} 2x_1 - 2 \\ 2x_2 \\ -2x_3 + 4 \end{pmatrix}$ $\qquad \nabla^2 f(x) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

- $x_N = (x_2 \quad x_3)$ $\quad x_B = x_1$ $\qquad N = (-1 , 2)$ $\quad B = 1 = B^{-1}$

- $Z = \begin{pmatrix} -B^{-1}N \\ \underbrace{I}_{} \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$

  $I : (n-m) \times (n-m)$

- $\underbrace{Z^T \nabla f(x)}_{\text{Reduced gradient}} = \begin{pmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2x_1 - 2 \\ 2x_2 \\ -2x_3 + 4 \end{pmatrix} = \begin{pmatrix} 2x_1 - 2 + 2x_2 \\ -4x_1 + 4 - 2x_3 + 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$2x_1 + 2x_2 = 2 \quad \div 2$

$$\left\{\begin{array}{l} 2x_1 - 2 + 2x_2 = 0 \\ -4x_1 + 4 - 2x_3 + 4 = 0 \\ x_1 - x_2 + 2x_3 = 2 \end{array}\right.$$

$\Big\}$ $z^T \nabla f(x) = 0$.

$-4x_1 - 2x_3 = -8 \quad \div(-2)$

• Feas. $Ax = b$

add the constraint as another eq-n

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 4 \\ 1 & -1 & 2 & 2 \end{array}\right) \begin{array}{c} -2R_1 \\ \sim \\ -R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -2 & 1 & 2 \\ 0 & -2 & 2 & 1 \end{array}\right) \sim \begin{array}{c} \div(-2) \\ -R_2 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & -1/2 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 5/2 \\ 0 & 1 & 0 & -3/2 \\ 0 & 0 & 1 & -1 \end{array}\right) \begin{array}{c} -R_2' \\ +\frac{1}{2}R_3 \end{array}$$

$$x^* = \begin{pmatrix} 5/2 \\ -3/2 \\ -1 \end{pmatrix} \rightarrow$$ the only point where $z^T \nabla f = 0$. and $Ax = b$

Now find $z^T \nabla^2 f z$

$$\underbrace{\begin{pmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}}_{z^T} \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}}_{\nabla^2 f} \underbrace{\begin{pmatrix} 1 & -2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}}_{z} =$$

$$= \begin{pmatrix} 2 & 2 & 0 \\ -4 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -4 \\ -4 & 6 \end{pmatrix} = z^T \nabla^2 f(x^*) z$$

Optimality conditions - example

- Solve the resultant system of equations:

- $x^* =$ Check second order condition (iii) for x^*:

$$Z^T \nabla^2 f(x^*) Z = \begin{pmatrix} 4 & -4 \\ -4 & 6 \end{pmatrix} \rightarrow \text{solve} \quad \det\left(\nabla^2 \Phi - \lambda I\right) = 0$$

with eigenvalues:

$$\det \begin{pmatrix} 4-\lambda & -4 \\ -4 & 6-\lambda \end{pmatrix} = 0$$

$$(4-\lambda)(6-\lambda) - 16 = 0$$

$$\lambda^2 - 10\lambda + 8 = 0$$

$$\lambda_{1,2} = \frac{10 \pm \sqrt{10^2 - 32}}{2} =$$

$$= \frac{10 \pm \sqrt{68}}{2}$$

both are

pos-ve

Hence $z^T \nabla^2 f(x^*) z$ is pos. - def $\rightarrow$

x^* is local min of $f(x)$

s.t. $Ax = b$.

Lagrangian function – preliminaries

- Let x^* a local minimiser of $f(x)$ over the set $\{x : Ax = b\}$, and Z is a basis matrix for the null-space of A. Then $\nabla f(x^*) = Zv^* + A^T\Lambda^*$. Hence

$$\nabla f(x^*) = \overset{\overset{\displaystyle p^*}{\parallel}}{p} + \overset{\overset{\displaystyle q^*}{\parallel}}{q} = Zv^* + A^T\Lambda^*$$

as $N(A)$ and $R(A^T)$ are orthogonal spaces

at min of $f(x)$ x^* $Z^T\nabla f(x^*) = 0 \rightarrow$

Hence $Z^T\nabla f(x^*) = \underbrace{Z^T Z v^*}_{} + Z^T \underbrace{A^T\Lambda^*}_{} = 0.$

$$Z^T Z v^* = 0 \qquad\qquad 0$$

$$\downarrow$$

$$v^* = 0$$

$$\downarrow$$

$$\nabla f(x^*) = A^T\Lambda^*$$

(Lambda capital)

where $\Lambda = (\lambda_1, \dots, \lambda_m)$ is a vector of Lagrangian multipliers

Lagrangian function – equality constraints

- Consider an NLP problem

$$\min z = f(x)$$

$$s.t. \quad g_i(x) = b_i, i = 1..m \qquad (**)$$

$Ax = b$ if linear constraints

- Introduce the Lagrangian function with Lagrangian multipliers $\Lambda = (\lambda_1, \dots, \lambda_m)$

$$L(x, \Lambda) = f(x) + \lambda^T (b - Ax) =$$

$$= f(x) + \sum_{i=1}^{m} \lambda_i (b_i - g_i(x))$$

$$L(x,\lambda) = f(x) + \sum_{i=1}^{m} \lambda_i (b_i - g_i(x))$$

$$\frac{\partial L(x,\lambda)}{\partial \lambda_i} = b_i - g_i(x)$$

Lagrangian function – equality con

- Assume that (x^*, Λ^*) minimazes $L(x, \Lambda)$. Then at (x^*, Λ^*)

$$\frac{\partial L(x,\Lambda)}{\partial \lambda_i} = \quad b_i - g_i(x^*) \quad = 0, i = 1..m$$

Hence x^* does/does not satisfy (**). → x^* is feasible for original problem (**)

To show that x^* is optimal , consider any feasible x' (for original problem)

$$L(x^*, \Lambda^*) = f(x^*) + \sum \lambda_i^* \underbrace{(b_i - g_i(x^*))}_{=0} \leq$$

$$\leq L(x', \Lambda') = f(x') + \sum \lambda_i' \underbrace{(b_i - g_i(x'))}_{=0}$$

as x' is feasible

$$\Bigg\} \; f(x^*) \leq f(x')$$

Summary: If (x^*, Λ^*) minimazes $L(x, \Lambda)$, then x^* is local min

of constrained NLP

min $f(x)$

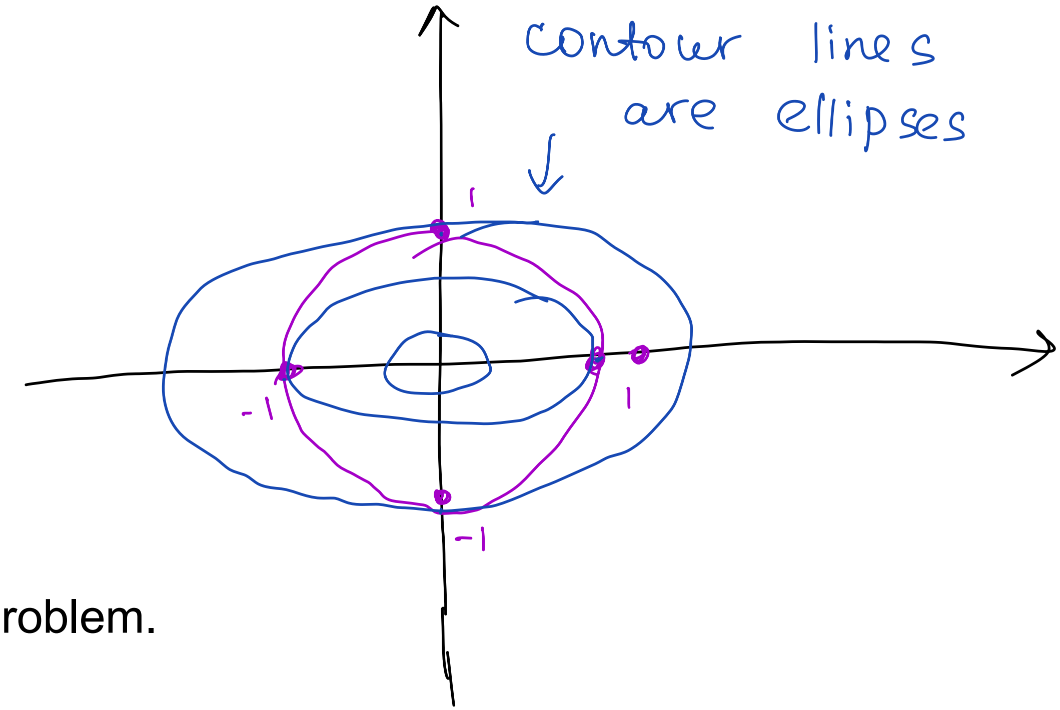
s.t. $Ax = b$

Example 1

Consider the NLP problem:

➢ $\min f(x_1, x_2) = x_1^2 + 2x_2^2$

$$s.t.\ \ x_1^2 + x_2^2 = 1$$



a) Write the Lagrangian function for this problem.

b) Use the Lagrangian to find local minimiser(s) for the given problem

Example 1

Consider the NLP problem:

- $\min f(x_1, x_2) = x_1^2 + 2x_2^2$

  $s.t.\ \ x_1^2 + x_2^2 = 1$

- $L(x_1, x_2, \lambda) = f(x) + \sum \lambda_i \left(b_i - g_i(x)\right) = x_1^2 + 2x_2^2 + \lambda(1 - x_1^2 - x_2^2)$

To find $\min L(x, \lambda)$

$\downarrow$

- $\nabla L(x_1, x_2, \lambda) = 0 \quad \Longrightarrow \quad \begin{array}{l} \text{w.r.t} \\ x_1 \\ x_2 \\ \lambda \end{array} \left\{ \begin{array}{l} \\ 2x_1 - 2x_1\lambda = 0 \\ 4x_2 - 2x_2\lambda = 0 \\ 1 - x_1^2 - x_2^2 = 0 \end{array} \right. \quad \rightarrow$

$$\rightarrow \begin{cases} 2x_1 = 2x_1\lambda & ① \\ 2x_2 = x_2\lambda & ② \\ x_1^2 + x_2^2 = 1 & ③ \end{cases}$$

① $x_1 = 0$ or $\lambda = 1 \quad (x_1 \neq 0)$

$\downarrow$

$x_2 = \pm 1$ (from ③)

$\downarrow$

$x_2 = \pm 1 \rightarrow \lambda = 2$

$(0, 1)$

$(0, -1)$

$\lambda = 1$

$\downarrow$

$x_2 = 0$ (from ②)

$\downarrow$

$x_1 = \pm 1$ (from ③)

$(-1, 0)$

$(1, 0)$

② $x_2 = 0$ or $\lambda = 2 \quad (x_2 \neq 0)$

$\downarrow$

$x_1 = \pm 1$

$\downarrow$

$\lambda = 1$

$\lambda = 2$

$\downarrow$

$x_1 = 0$ (from (1))

$\downarrow$

$x_2 = \pm 1$

$$L(\underbrace{\pm 1, 0, 1}_{\text{min}}) = 1 + 0 + 1 \times (1 - 1 - 0) = 1$$

$$L(x, \lambda) = x_1^2 + 2x_2^2 + \lambda(1 - x_1^2 - x_2^2)$$

$$L(\underbrace{0, \pm 1, 2}_{\text{max}}) = 0 + 2 + 2 \times (1 - 0 - 1) = 2$$

Hence $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $x = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ are local minimisers of $f(x_1, x_2)$ on $\{x : x_1^2 + x_2^2 = 1\}$

Lagrangian function – equality constraints

- The first-order optimality condition for unconstrained NLP requires that

$\nabla L(x,\Lambda) =$ 0 $\quad$ $i.e.\ \nabla_\Lambda L(x,\Lambda) =$ $b - Ax$ $= 0$ and $\nabla_x L(x,\Lambda) =$ $= \nabla f(x) - A^T\lambda = 0$ (***)

$\nabla_x L(x,\Lambda) = \quad \Leftrightarrow \quad \nabla f(x) =$

$\nabla f(x) = A^T\lambda.$

- Any point (x', Λ') satisfying (***) is a s……tationary……point for $L(x,\Lambda)$ and a feasible point for (**).

• Statitionary point of $L(x,\lambda)$ (x^*, λ^*) : x^* is feasible for constrained opt. problem (**)

Lagrangian function – equality constraints

Theorem 3. - constrained

➢ If (x^*, Λ^*) is a stationary point to $L(x, \Lambda)$:

1. • $\frac{\partial L(x,\Lambda)}{\partial \lambda_i} = 0, i = 1..m$ → Feasibility of x^*

2. • $\frac{\partial L(x,\Lambda)}{\partial x_j} = 0, j = 1..n$ → $\underbrace{\nabla f(x^*) = A^T \Lambda^*}$ → $Z^T \nabla f(x^*) =$

$= Z^T A^T \Lambda^* = 0$

3. *Each* $g_i(x)$ is linear *__And__* *f(x)* is a convex function,

then x^* is a local minimum of *f(x) on* $\underbrace{\{g(x) = b\}}$

if $f(x)$ is not convex, then

check if $Z^T \nabla^2 f(x^*) Z$ is pos. - def

Example 2

Consider the NLP problem:

- $\min f(x_1, x_2, x_3) = x_1^2 - 2x_1 + x_2^2 - x_3^2 + 4x_3$

s.t. $\qquad x_1 - x_2 + 2x_3 = 2$

a) Write the Lagrangian function for this problem.

b) Use the Lagrangian to find local minimiser(s) for the given problem

Example 2

Consider the NLP problem:

- $\min f(x_1, x_2, x_3) = x_1^2 - 2x_1 + x_2^2 - x_3^2 + 4x_3$

s.t. $x_1 - x_2 + 2x_3 = 2$

$L(x_1, x_2, x_3, \lambda) = x_1^2 - 2x_1 + x_2^2 - x_3^2 + 4x_3 + \lambda(2 - x_1 + x_2 - 2x_3)$

1. Find stationary point of L

- $\nabla L(x_1, x_2, x_3, \lambda) = 0 \Rightarrow \begin{cases} 2x_1 - 2 - \lambda = 0 \\ 2x_2 + \lambda = 0 \\ -2x_3 + 4 - 2\lambda = 0 \\ 2 - x_1 + x_2 - 2x_3 = 0 \end{cases}$

$\rightarrow -2x_3 - 2\lambda = -4$

$(\div(-2))$

$x_1 - x_2 + 2x_3 = 2$

$$\left(\begin{array}{cccc|c} 2 & 0 & 0 & -1 & 2 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 \\ 1 & -1 & 2 & 0 & 2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1/2 & 1 \\ 0 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & -1 & 2 & 1/2 & 1 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1/2 & 1 \\ 0 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1/2 & 1 \\ 0 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & -1 & -3 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5/2 \\ 0 & 1 & 0 & 0 & -3/2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right) \to (x^*, \lambda^*)^T = \begin{pmatrix} 5/2 \\ -3/2 \\ -1 \\ 3 \end{pmatrix} \quad x^* = \begin{pmatrix} 5/2 \\ -3/2 \\ -1 \end{pmatrix}, \quad \lambda^* = 3$$

Assume we found (x^*, λ^*) $\to$ stationary point of $L(x, \lambda)$.

Check condition 3 of Th. 3 - constr.

$g(x) = b \to$ linear

$g(x) \to x_1 - x_2 + 2x_3 = 2$

$$\nabla f(x) = \begin{pmatrix} 2x_1 - 2 \\ 2x_2 \\ -2x_3 + 4 \end{pmatrix} ; \quad \nabla^2 f(x) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$\downarrow$

pos.-def for all $x \in R^n$

$\downarrow$

$f(x)$ is strictly convex

$\downarrow$

x^* is local min on $\{x : ax - b = 0\}$

Example 3*

in tutorial

Consider the NLP problem:

- $\min f(x_1, x_2, x_3) = 3x_1^2 - \frac{1}{2}x_2^2 - \frac{1}{2}x_3^2 + x_1x_2 - x_1x_3 + 2x_2x_3$

s.t. $2x_1 - x_2 + x_3 = 2$

a) Write the Lagrangian function for this problem.

b) Use the Lagrangian to find local minimiser(s) for the given problem

* from Linear and Non-Linear Programming by S.G.Nash and A.Sofer

Example 3

Consider the NLP problem:

- $\min f(x_1, x_2, x_3) = 3x_1^2 - \frac{1}{2}x_2^2 - \frac{1}{2}x_3^2 + x_1x_2 - x_1x_3 + 2x_2x_3$

s.t. $2x_1 - x_2 + x_3 = 2$

- $\nabla L(x_1, x_2, x_3, \lambda) = \quad \Rightarrow \quad \begin{cases} \\ \\ \end{cases}$