Introduction to Optimisation:

Duality of Linear Programming

Lectures 5-6

Lecture notes by Dr. Julia Memar and Dr. Hanyu Gu and with an acknowledgement to Dr.FJ Hwang and Dr.Van Ha Do

Duality of Linear Programming

Example 1: Firm A aims to produce 7kg of gold and 2kg of nickel to meet a contract. Each tonne of ore from mine 1 yields 2kg of gold and 1kg of nickel, whilst each tonne of ore from mine 2 yields 5kg of gold and 4kg of nickel. Mining one tonne from mine 1 costs $300, but costs $100 from mine 2. The objective of firm A is to minimise the cost of producing enough gold and nickel to meet the contract.

Summary:

	Gold (kg/tonne)	Nickel (kg/tonne)	Cost ($100/tonne)
Mine 1	2	1	3
Mine 2	5	4	1
Demand (kg)	7	2	

<u>Formulation:</u>

x_i - tonnes from mine i

OF : min $3x_1 + x_2$

s.t. $2x_1 + 5x_2 \geq 7$ - gold

$x_1 + 4x_2 \geq 2$ - nickel

$x_i \geq 0$.

$c^P = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ $b^P = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$ $A^P = \begin{pmatrix} 2 & 5 \\ 1 & 4 \end{pmatrix}$

Duality of Linear Programming

Example 1 : Now suppose that in the market there exists another firm B, that sells gold and nickel. Firm B happens to know everything about firm A's operation costs and tries to set the selling prices ($100/kg) y_1 and y_2 of gold and nickel, respectively, for firm A. The objective of firm B is to maximise the revenue from the sale of 7kg of gold plus 2kg of nickel. Firm B knows that firm A will dig out their own minerals if it is cheaper than buying off firm B. To avoid the situation that the selling price is higher than the cost of mining from mine 1 and 2, firm B shall set the price of 2kg of gold plus 1kg of nickel no greater than $300 and that of 5kg of gold plus 4kg of nickel no greater than $100.

<u>Formulation:</u> Let y_1 y_2 be prices per TONN of gold Nickel (in $100)

max $7y_1 + 2y_2$

M1 : $2y_1 + y_2 \leq 3$ → cost per TONN mine ≤ 300

M2 : $5y_1 + 4y_2 \leq 1$ →

$$c^D = \begin{pmatrix} 7 \\ 2 \end{pmatrix} \quad b^D = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad A^D = \begin{pmatrix} 2 & 1 \\ 5 & 4 \end{pmatrix}$$

$$c^P = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad b^P = \begin{pmatrix} 7 \\ 2 \end{pmatrix} \quad A^P = \begin{pmatrix} 2 & 5 \\ 1 & 4 \end{pmatrix}$$

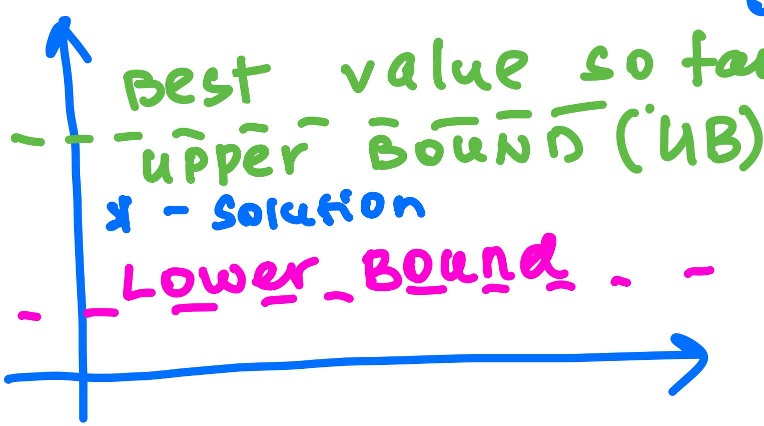
Observations:

$c^P = b^D$; $A^D = (A^P)^T$

$c^D = b^P$

Motivation

- Optimisation problems are complex
- Optimal solution is easy/hard/impossible to find
- Need to evaluate a solution: How? → use upper and lower bounds on the optimal value



Assume we are solving min problem

min z $z^* \leq z^{\text{best so far}}$

$$\frac{z^{\text{best s-n}} - LB}{LB}$$ relative optimality gap

Normal form of a linear program

The **normal form** of a linear program:

- $\min \; z = \boldsymbol{c}^T \boldsymbol{x}$

$$s.t. \quad \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}, \quad \boldsymbol{x} \geq \boldsymbol{0},$$

where b ***is not*** restricted in sign.

greater than

min normal form

OR

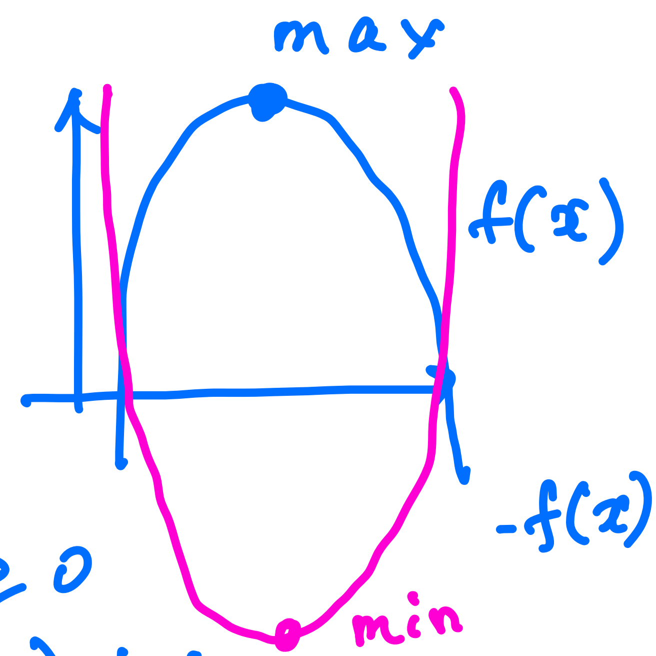
- $\max \; z = \boldsymbol{c}^T \boldsymbol{x}$

$$s.t. \quad \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \quad \boldsymbol{x} \geq \boldsymbol{0},$$

less than

max normal form

Normal form of a linear program



➢ Any LP can be transformed into the normal form:

- $\max z = \boldsymbol{c}^T\boldsymbol{x} \quad \Leftrightarrow \quad \min\ z' = -\boldsymbol{c}^T\boldsymbol{x}$

- If x_i urs $x_i = x_i' - x_{ii}''$, x_i' , $x_i'' \geq 0$
- To replace " ≤ " constraint by " ≥ " constraint $\times(-1)$ both sides

- Replace " = " constraint by " ≤ " *and* " ≥ " constraints

LHS = RHS

↓

LHS ≤ RHS

LHS ≥ RHS

Example 1

Consider the LP:

- $\max z = 3x_1 + 2x_2$

  s.t. $2x_1 + 5x_2 = 7$

  $x_1 + x_2 \geq 3$

  $x_1,\ x_2 \geq 0$

$\begin{cases} 2x_1 + 5x_2 \geq 7 \\ 2x_1 + 5x_2 \leq 7 \end{cases}$

not normal

$\times (-1)$

- Normal ***minimization*** form:

  $\min z' = -3x_1 - 2x_2$

  s.t.

  $2x_1 + 5x_2 \geq 7$

  $-2x_1 - 5x_2 \geq -7$

  $x_1 + x_2 \geq 3$

  $x_1,\ x_2 \geq 0$

- Normal ***maximization*** form:

  $\max z = 3x_1 + 2x_2$

  s.t.

  $2x_1 + 5x_2 \leq 7$

  $-2x_1 - 5x_2 \leq -7$

  $-x_1 - x_2 \leq -3$

  $x_1,\ x_2 \geq 0$

Dual LP

For any LP there exists a dual LP:

- $\min\ z = \boldsymbol{c}^{T}\boldsymbol{x}$

$$s.t.\quad \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}, \qquad (\boldsymbol{P})$$

$$\boldsymbol{x} \geq \boldsymbol{0}$$

primal

$x_j \in x, j = 1..n$ - primal variable, associated with jth constraint in dual problem

- $\max\ w = \boldsymbol{b}^{T}\boldsymbol{y}$

$$s.t.\quad \boldsymbol{A}^{T}\boldsymbol{y} \leq \boldsymbol{c}, \qquad (\boldsymbol{D})$$

$$\boldsymbol{y} \geq \boldsymbol{0}$$

dual

$y_i \in y, i = 1..m$ - dual variable, associated with ith constraint in primal problem

Duality of Linear Programming

Example 2 - Primal LP: A baker makes and sells two types of cakes, one is a simple cake and another is a fancy cake. Both cakes require basic ingredients (flour, sugar, eggs, etc) as well as premium ingredients such as nuts and fruits for decoration and flavour, with the fancy cake requiring more of the premium ingredients. The fancy cake production also require higher labour costs. The baker needs to maximize the profit.

Summary:

	Basic ingredients per batch of cakes, lb	Premium ingredients per batch of cakes, lb	Labour, per batch of cakes, hours	Profit per batch, $
Simple cake	2	1	1	$24
Fancy cake	3	4	2	$14
Available ingredients	1200	1000	700	

Duality of Linear Programming

Example 2 - Primal LP:

Summary:

	Basic ingredients per batch of cakes, lb	Premium ingredients per batch of cakes, lb	Labour, per batch of cakes, hours	Profit per batch, $
Simple cake	2	1	1	$24
Fancy cake	3	4	2	$14
Available ingredients	1200	1000	700	

Formulation:

x_i - n of simple (x_1) or fancy (x_2) cakes

OF: max $\$24x_1 + \$14x_2$

s.t.

$$2x_1 + 3x_2 \le 1200$$
$$x_1 + 4x_2 \le 1000$$
$$x_1 + 2x_2 \le 700$$
$$x_1, x_2 \ge 0$$

(P) in normal max form

Duality of Linear Programming

Example 2 - Dual LP: formulate the dual problem and provide the interpretation of the dual variables, objective function and constraints

(P)

$$\max \ \$24x_1 + \$14x_2$$

s.t.

$$2x_1 + 3x_2 \le 1200$$
$$x_1 + 4x_2 \le 1000$$
$$x_1 + 2x_2 \le 700$$
$$x_1 \ x_2 \ge 0$$

$\rightarrow$

$$\min \ -24x_1 - 14x_2$$

s.t.

$$-2x_1 - 3x_2 \ge -1200$$
$$-x_1 - 4x_2 \ge -1000$$
$$-x_1 - 2x_2 \ge -700$$
$$x_1, x_2$$

(D)

$$\max \ -1200y_1 - 1000y_2 - 700y_3$$

s.t.

$$-2y_1 - y_2 - y_3 \le -24$$
$$-3y_1 - 4y_2 - 2y_3 \le -14$$
$$y_1 \ y_2 \ y_3 \ge 0$$

$\rightarrow$

$$\min \ 1200y_1 + 1000y_2 + 700y_3$$

s.t.

$$2y_1 + y_2 + y_3 \ge 24$$
$$3y_1 + 4y_2 + 2y_3 \ge 14$$
$$y_1, y_2 \ y_3 \ge 0$$

Dual LP

	$\mathbf{x}^T$	
$\mathbf{y}$	$\mathbf{A}$	$\geq \mathbf{b}$
	$\leq \mathbf{c}^T$	

(P)

D	x_1	x_2	$\cdots$	x_n	
y_1	a_{11}	a_{12}	$\cdots$	a_{1n}	$\geq b_1$
y_2	a_{21}	a_{22}	$\cdots$	a_{2n}	$\geq b_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
y_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}	$\geq b_m$
	$\leq c_1$	$\leq c_2$	$\cdots$	$\leq c_n$	

In matrix form:

If LP is in normal form

- Symmetric dual

Otherwise

- Asymmetric dual:
  - Bring primal (P) to normal form (P')
  - Construct dual (D) for (P')

Example 2

- Primal LP:

$$\min z = 3x_1 + x_2$$
$$\text{s.t.}\quad 2x_1 + 5x_2 \ge 7$$
$$x_1 + 4x_2 \ge 2$$
$$x_1,\ x_2 \ge 0$$

→ Normal Form

- Dual LP:

$$\max w = 7y_1 + 2y_2$$
s.t.
$$2y_1 + y_2 \le 3$$
$$5y_1 + 4y_2 \le 1$$
$$y_1\ y_2 \ge 0$$

Example 3

$$\min z = 4x_1 + 12x_2 + x_3$$

s.t.

$$-x_1 - 4x_2 + x_3 \leq 1$$
$$2x_1 + 2x_2 + x_3 = 2$$
$$x_1, x_2 \geq 0 ; \quad x_3 - \text{URS}$$

(P) — not in normal form: $(\times -1)$, $\rightarrow \leq \geq$ form

Let $x_3 = x_3' - x_3''$

$$\min z = 4x_1 + 12x_2 + x_3' - x_3''$$

s.t.

$$x_1 + 4x_2 \underbrace{- x_3' + x_3''}_{-x_3} \geq -1$$
$$2x_1 + 2x_2 + x_3' - x_3'' \geq 2$$
$$-2x_1 - 2x_2 - x_3' + x_3'' \geq -2$$
$$x_1, x_2, x_3', x_3'' \geq 0$$

(P) in normal form

$$\max -y_1 + 2y_2 - 2y_3$$

s.t.

$$y_1 + 2y_2 - 2y_3 \leq 4$$
$$4y_1 + 2y_2 - 2y_3 \leq 12$$
$$-y_1 + y_2 - y_3 \leq 1$$
$$y_1 - y_2 + y_3 \leq -1$$
$$y_1, y_2, y_3 \geq 0.$$

Dual in normal form

$$\max \quad -y_1 + 2y_2 - 2y_3$$

$$\text{s.t.} \quad \begin{aligned} y_1 + 2y_2 - 2y_3 &\le 4 \\ 4y_1 + 2y_2 - 2y_3 &\le 12 \\ -y_1 + y_2 - y_3 &\le 1 \\ y_1 - y_2 + y_3 &\le -1 \\ y_1, y_2, y_3 &\ge 0. \end{aligned}$$

$$\begin{aligned} -y_1 + y_2 - y_3 &\le 1 \\ -y_1 + y_2 - y_3 &\ge 1 \\ \hline -y_1 + y_2 - y_3 &= 1 \end{aligned}$$

Let $y_2 - y_3 = \tilde{y} \rightarrow \tilde{y}$ - URS

$\bar{y} = -y_1 \rightarrow \bar{y} \le 0$

$$\max \quad \bar{y} + 2\tilde{y}$$

$$\text{s.t.} \quad \begin{aligned} -\bar{y} + 2\tilde{y} &\le 4 \\ -4\bar{y} + 2\tilde{y} &\le 12 \\ \bar{y} + \tilde{y} &= 1 \end{aligned}$$

$\bar{y} \le 0$, $\tilde{y}$ - URS

$$\min z = 4x_1 + 12x_2 + x_3$$

s.t. $-x_1 - 4x_2 + x_3 \le 1$

$2x_1 + 2x_2 + x_3 = 2$

$x_1, x_2 \ge 0; \; x_3$ - URS

not in normal form: $(\times -1)$ $\to \le, \ge$

(P)

$$\max \; \bar{y} + 2\tilde{y}$$

s.t. $-\bar{y} + 2\tilde{y} \le 4$

$-4\bar{y} + 2\tilde{y} \le 12$

$\bar{y} + \tilde{y} = 1$

$\bar{y} \le 0, \; \tilde{y}$ - URS

dual not in normal form

Primal

1st primal constr $(\times -1) \to \bar{y} \le 0$

2nd primal constr $= \;\to \tilde{y}$ - URS

Dual

x_3 is URS $\to$ 3rd dual constraint "="

$\bar{y} \le 0 \;\to$ "reversed" 1st dual constraint

HW

Example 4*

- **Primal LP:**

$$\max z = 6x_1 + x_2 + x_3$$

s.t.

$$4x_1 + 3x_2 - 2x_3 = 1$$
$$6x_1 - 2x_2 + 9x_3 \geq 9$$
$$2x_1 + 3x_2 + 8x_3 \leq 5$$
$$x_1 \geq 0,\ \ x_2 \leq 0, x_3 - urs$$

- Dual LP:

* from Linear and Non-Linear Programming by S.G.Nash and A.Sofer

Example 4*

* from Linear and Non-Linear Programming by S.G.Nash and A.Sofer

Dual LP

	x_1	x_2	$\cdots$	x_n	
y_1	a_{11}	a_{12}	$\cdots$	a_{1n}	$\geq b_1$
y_2	a_{21}	a_{22}	$\cdots$	a_{2n}	$\geq b_2$
y_i					$\leq b_i$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
y_j					$= b_j$
y_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}	$\geq b_m$
	$\leq c_1$	$\leq c_2$	$\cdots$	$\leq c_n$	

- Asymmetric dual:

apply the following rules:

primal/dual constraint		**dual/primal variable**
consistent with normal form	$\Longleftrightarrow$	variable ≥ 0
reversed with normal form	$\Longleftrightarrow$	variable ≤ 0
equality constraint	$\Longleftrightarrow$	variable urs

hw

Example 4*

- **Primal LP:**

$$\max z = 6x_1 + x_2 + x_3$$

s.t.

$$4x_1 + 3x_2 - 2x_3 = 1$$
$$6x_1 - 2x_2 + 9x_3 \geq 9$$
$$2x_1 + 3x_2 + 8x_3 \leq 5$$
$$x_1 \geq 0,\ \ x_2 \leq 0, x_3 - urs$$

- Dual LP:

* from Linear and Non-Linear Programming by S.G.Nash and A.Sofer

Duality theory

Lemma 1 The dual of the dual is the primal.

Example:

- Primal LP:

$$\min z = 3x_1 + x_2$$

s.t. $2x_1 + 5x_2 \geq 7$

$x_1 + 4x_2 \geq 2$

$x_1, \; x_2 \geq 0$

- Dual LP:

Show that dual of the dual is the primal problem

Weak duality theorem

For a feasible solution $\boldsymbol{x}$ to the primal LP (P) and a feasible solution $\boldsymbol{y}$ to the dual LP (D),

$$\boldsymbol{c}^T\boldsymbol{x} \geq \boldsymbol{b}^T\boldsymbol{y}$$

In other words, $z \geq w$.

Example

- Primal LP:

$$\min z = 3x_1 + x_2$$

$$\text{s.t.}\quad 2x_1 + 5x_2 \geq 7$$

$$x_1 + 4x_2 \geq 2$$

$$x_1,\ x_2 \geq 0$$

- Dual LP:

Compare values of Z and W for any two feasible solutions for primal and dual problems

Weak duality theorem

Weak duality theorem – optimality gap

- Primal LP:

$\min z = 5x_1 + 2x_2$

s.t. $x_1 - x_2 \geq 3$

$2x_1 + 3x_2 \geq 5$

$x_1,\ x_2 \geq 0$

- Dual LP:

- How can we estimate ***the optimality gap*** based on the weak duality theorem?

Weak duality theorem

Corollary 3

1) If the primal LP is unbounded, then the dual LP is infeasible.
2) If the dual LP is unbounded, then the primal LP is infeasible.

Example

$$\max z = 2x_1 + 3x_2$$

s.t.

$$x_1 - x_2 \le 1$$

$$x_1 - 2x_2 \le 2$$

$$x_1,\ x_2 \ge 0$$

Corollary - example

Strong duality theorem

Let x be a feasible solution for primal LP (P) and y be a feasible solution of the corresponding dual LP (D). Then

$$\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{b}^T\boldsymbol{y}$$

if and only if x is an optimal solution for (P) and y is an optimal solution for (D).

(We need the dual of the LP in the standard form)

Corollary

If an optimal solution to the dual LP (D) is obtained, an optimal solution to its primal LP (P) can be readily obtained and both optimal objective values are equal.

Strong duality theorem

Example : solve primal /dual of Example 2, apply the strong duality theorem to the corresponding dual/primal problem to find its optimal solution.

Strong duality theorem

Summary

Dual

Primal

	Finite optimum	**unbounded**	**infeasible**
Finite optimum			
unbounded			
infeasbile			

Complementary slackness theorem

Theorem 6

- Let x be a feasible solution to the primal LP (P) and y be a feasible solution to the dual LP (D). Both solutions x and y are optimal to the primal (P) and dual (D), respectively, if and only if they satisfy

$$(\boldsymbol{c}^T - \boldsymbol{y}^T\boldsymbol{A})\boldsymbol{x} = \boldsymbol{0} \quad \textit{and} \quad \boldsymbol{y}^T(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}) = \boldsymbol{0} \qquad (2)$$

Complementary slackness theorem

- **Example 7** : consider the LP

$\min z = 2x_1 + 2x_2$

s.t. $2x_1 + x_2 \geq 6$

$x_1 + 2x_2 \geq 6$

$x_1, \ x_2 \geq 0$

Solve the dual problem and, with the Complementary slackness Theorem, obtain the solution for the primal problem.

Complementary slackness theorem

* from Linear and Non-Linear Programming by S.G.Nash and A.Sofer

Dual Simplex method - summary

- *Primal Simplex* – finding series of solutions satisfying *primal* feasibility condition

$$x_b = B^{-1}b \geq 0$$

till finds a solution satisfying *primal* optimality condition:

$$\widehat{c_N^T} \leq 0 \qquad \text{(if primal is a min problem),}$$

- *Dual Simplex* – finding series of solutions satisfying *dual* feasibility condition till finds a solution satisfying *dual* optimality condition:

$$\widehat{b_N^T} \geq 0 \text{(dual is a max problem then)}$$

- *Tableau:* working out in reverse order – finding the leaving variable first.
- *Use for*

  1) generating initial bfs and

  2) recovering feasibility

Dual Simplex method

- Step 1 (feasibility test) If $\hat{b} = B^{-1}\, b \geq 0$, then STOP – current primal bfs is feasible;

Otherwise, select the variable $(x_B)_s$ whose rhs $\hat{b}_s$ is the most negative among components of $\hat{b}$. This component of the primal basis will be *leaving*

- Step 2 (entering) If all entries $\widehat{a_{sj}}$ in the row corresponding to the leaving variable$(x_B)_s$ are non-negative ($\widehat{a_{sj}} \geq 0$, for any $j = 1,2, \ldots, n$), then STOP – the considered LP (the primal) is infeasible. Otherwise, find

$$t = \arg\min_j \left\{ \left| \frac{\widehat{c_j}}{\widehat{a_{sj}}} \right| : \widehat{a_{sj}} < 0 \right\}$$

- Step 3 (pivoting) Update the tableau by pivoting on $\widehat{a_{sj}}$, i.e. perform EROs on the tableau to get a 1 in the pivot position, and 0s above and below it. GO TO Step 1.

Generating initial bfs

- **Example 9**

$$\max z = 4x_1 + 5x_2$$

s.t. $2x_1 + 3x_2 \le 6$

$3x_1 + x_2 \ge 3$

$x_1,\ x_2 \ge 0$

Recovering the Feasibility

Example 10

$$\max z = 0.1x_1 + 0.15x_2$$

s.t.

$$x_1 + x_2 \leq 100000$$

$$-\frac{3}{4}x_1 + \frac{1}{4}x_2 \leq 0$$

$$-\frac{3}{2}x_1 + x_2 \leq 0$$

$$x_1,\ x_2 \geq 0$$

Dual Simplex method – examples*

Use dual Simplex method to solve:

$$\min\ z = 5x_1 + 4x_2$$

s.t. $4x_1 + 3x_2 \geq 10$

$3x_1 - 5x_2 \geq 12$

$x_1,\ x_2 \geq 0$

* from Linear and Non-Linear Programming by S.G.Nash and A.Sofer

Dual Simplex method – examples*

Use dual Simplex method to solve:

$$\max\ z = -2x_1 - 7x_2 - 6x_3 - 5x_4$$

s.t.
$$2x_1 - 3x_2 - 5x_3 - 4x_4 \geq 20$$
$$7x_1 + 2x_2 + 6x_3 - 2x_4 \geq 35$$
$$4x_1 + 5x_2 - 3x_3 - 2x_4 \geq 15$$
$$x_1,\ x_2,\ x_3, x_4 \geq 0$$

* from Linear and Non-Linear Programming by S.G.Nash and A.Sofer