

Introduction to Optimisation:

Duality of Linear Programming

Lectures 5-6

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Duality of Linear Programming

Example 1: Firm A aims to produce 7kg of gold and 2kg of nickel to meet a contract. Each tonne of ore from mine 1 yields 2kg of gold and 1kg of nickel, whilst each tonne of ore from mine 2 yields 5kg of gold and 4kg of nickel. Mining one tonne from mine 1 costs \$300, but costs \$100 from mine 2. The objective of firm A is to minimise the cost of producing enough gold and nickel to meet the contract.

Summary:

	Gold (kg/tonne)	Nickel (kg/tonne)	Cost (\$100/tonne)
Mine 1	2	1	3
Mine 2	5	4	1
Demand (kg)	7	2	

Formulation:

x_i - tonnes from mine i

$$\text{OF : } \min 3x_1 + x_2$$

$$\text{s.t. } \begin{aligned} 2x_1 + 5x_2 &\geq 7 && - \text{gold} \\ x_1 + 4x_2 &\geq 2 && - \text{nickel} \end{aligned}$$

$$x_i \geq 0.$$



$$c^P = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad b^P = \begin{pmatrix} 7 \\ 2 \end{pmatrix} \quad A^P = \begin{pmatrix} 2 & 5 \\ 1 & 4 \end{pmatrix}$$

Duality of Linear Programming

Example 1 : Now suppose that in the market there exists another firm B, that sells gold and nickel. Firm B happens to know everything about firm A's operation costs and tries to set the selling prices (\$100/kg) y_1 and y_2 of gold and nickel, respectively, for firm A. The objective of firm B is to maximise the revenue from the sale of 7kg of gold plus 2kg of nickel. Firm B knows that firm A will dig out their own minerals if it is cheaper than buying off firm B. To avoid the situation that the selling price is higher than the cost of mining from mine 1 and 2, firm B shall set the price of 2kg of gold plus 1kg of nickel no greater than \$300 and that of 5kg of gold plus 4kg of nickel no greater than \$100.

Formulation: let y_1 y_2 be prices per TONN of
gold Nickel (in \$100)

$$\max \quad 7y_1 + 2y_2$$

$$M1 : \quad 2y_1 + y_2 \leq 3 \quad \rightarrow \left. \begin{array}{l} \text{cost per TONN} \\ \text{mine} \leq 300 \end{array} \right\}$$

$$M2 : \quad 5y_1 + 4y_2 \leq 1 \quad \rightarrow$$

$$c^D = \begin{pmatrix} 7 \\ 2 \end{pmatrix} \quad b^D = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad A^D = \begin{pmatrix} 2 & 1 \\ 5 & 4 \end{pmatrix}$$

$$c^P = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad b^P = \begin{pmatrix} 7 \\ 2 \end{pmatrix} \quad A^P = \begin{pmatrix} 2 & 5 \\ 1 & 4 \end{pmatrix}$$

Observations:

$$c^P = b^D ; \quad A^D = (A^P)^T$$
$$c^D = b^P$$

Motivation

➤ Optimisation problems are complex

➤ Optimal solution is easy/hard/impossible to find

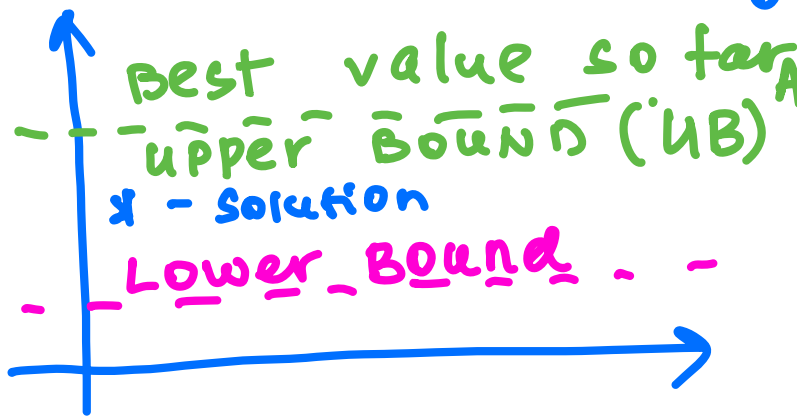
➤ Need to evaluate a solution: How?

→ use upper and lower bounds on the optimal value

Assume we are solving

min problem

$$\boxed{\min z} \quad z^* \leq z_{\text{so far}}^{\text{best}}$$



$$\frac{\text{Best s-n } z}{LB} - LB$$

relative optimality gap

Normal form of a linear program

The **normal form** of a linear program:

$$\triangleright \min z = c^T x$$

greater than

$$\text{s.t. } Ax \geq b, \\ x \geq 0,$$

where b **is not** restricted in sign.

min normal form

OR

$$\triangleright \max z = c^T x$$

less than

$$\text{s.t. } Ax \leq b, \\ x \geq 0,$$

max normal form

Normal form of a linear program

➤ Any LP can be transformed into the normal form:

- $\max z = c^T x \Leftrightarrow \min z' = -c^T x$

- If x_i unbounded $x_i = x_i' - x_i''$, $x_i', x_i'' \geq 0$

- To replace " \leq " constraint by " \geq " constraint $\times (-1)$ both sides

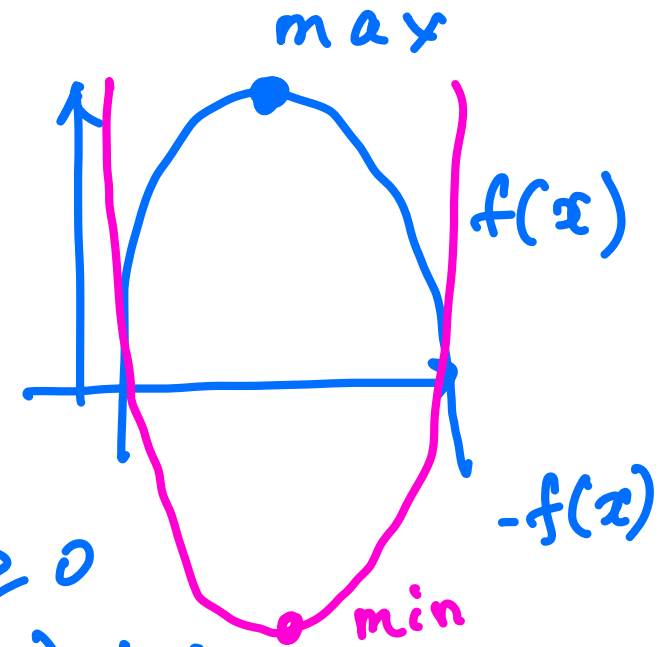
- Replace " $=$ " constraint by " \leq " and " \geq " constraints

$$\text{LHS} = \text{RHS}$$



$$\text{LHS} \leq \text{RHS}$$

$$\text{LHS} \geq \text{RHS}$$



Example 1

Consider the LP:

$$\text{max } z = 3x_1 + 2x_2$$

$$\text{s.t. } 2x_1 + 5x_2 = 7$$

$$x_1 + x_2 \geq 3$$

$$x_1, x_2 \geq 0$$

$$\begin{cases} 2x_1 + 5x_2 \geq 7 \\ 2x_1 + 5x_2 \leq 7 \end{cases}$$

} not normal
* (-1)

➤ Normal **minimization** form:

$$\min z' = -3x_1 - 2x_2$$

s.t.

$$\begin{aligned} 2x_1 + 5x_2 &\geq 7 \\ -2x_1 - 5x_2 &\geq -7 \\ x_1 + x_2 &\geq 3 \end{aligned}$$

$$x_1, x_2 \geq 0$$

➤ Normal **maximization** form:

$$\max z = 3x_1 + 2x_2$$

s.t.

$$\begin{aligned} 2x_1 + 5x_2 &\leq 7 \\ -2x_1 - 5x_2 &\leq -7 \\ -x_1 - x_2 &\leq -3 \end{aligned}$$

$$x_1, x_2 \geq 0$$

Dual LP

For any LP there exists a dual LP:

$$\begin{aligned} \triangleright \min z &= \mathbf{c}^T \mathbf{x} \\ \text{s.t. } \quad & \mathbf{Ax} \geq \mathbf{b}, \quad (P) \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \quad \text{primal}$$

$x_j \in \mathbf{x}, j = 1..n$ - primal variable,
associated with j th constraint in
dual problem

$$\begin{aligned} \triangleright \max w &= \mathbf{b}^T \mathbf{y} \\ \text{s.t. } \quad & \mathbf{A}^T \mathbf{y} \leq \mathbf{c}, \quad (D) \\ & \mathbf{y} \geq \mathbf{0} \end{aligned} \quad \text{dual}$$

$y_i \in \mathbf{y}, i = 1..m$ - dual variable,
associated with i th constraint in primal
problem

Duality of Linear Programming

Example 2 - Primal LP: A baker makes and sells two types of cakes, one is a simple cake and another is a fancy cake. Both cakes require basic ingredients (flour, sugar, eggs, etc) as well as premium ingredients such as nuts and fruits for decoration and flavour, with the fancy cake requiring more of the premium ingredients. The fancy cake production also require higher labour costs. The baker needs to maximize the profit.

Summary:

	Basic ingredients per batch of cakes, lb	Premium ingredients per batch of cakes, lb	Labour, per batch of cakes, hours	Profit per batch, \$
Simple cake	2	1	1	\$24
Fancy cake	3	4	2	\$14
Available ingredients	1200	1000	700	

Duality of Linear Programming

Example 2 - Primal LP:

Summary:

	Basic ingredients per batch of cakes, lb	Premium ingredients per batch of cakes, lb	Labour, per batch of cakes, hours	Profit per batch, \$
Simple cake	2	1	1	\$24
Fancy cake	3	4	2	\$14
Available ingredients	1200	1000	700	

Formulation:

x_i - n of simple or fancy cakes
 x_1 x_2

OF: s.t.

(P) in normal max form

$$\begin{aligned} \max & \quad \$24x_1 + \$14x_2 \\ \text{s.t.} & \quad 2x_1 + 3x_2 \leq 1200 \\ & \quad x_1 + 4x_2 \leq 1000 \\ & \quad x_1 + 2x_2 \leq 700 \\ & \quad x_1, x_2 \geq 0 \end{aligned}$$

Duality of Linear Programming

Example 2 - Dual LP: formulate the dual problem and provide the interpretation of the dual variables, objective function and constraints

$$\begin{aligned} & \max \quad \$24x_1 + \$14x_2 \\ \text{s.t.} \quad & 2x_1 + 3x_2 \leq 1200 \\ (P) \quad & x_1 + 4x_2 \leq 1000 \\ & x_1 + 2x_2 \leq 700 \\ & x_1, x_2 \geq 0 \end{aligned} \quad \rightarrow$$

$$\begin{aligned} & \min \quad -24x_1 - 14x_2 \\ \text{s.t.} \quad & -2x_1 - 3x_2 \geq -1200 \\ & -x_1 - 4x_2 \geq -1000 \\ & -x_1 - 2x_2 \geq -700 \\ & x_1, x_2 \end{aligned}$$

$$\begin{aligned} & \max \quad -1200y_1 - 1000y_2 - 700y_3 \\ \text{s.t.} \quad & -2y_1 - y_2 - y_3 \leq -24 \\ (D) \quad & -3y_1 - 4y_2 - 2y_3 \leq -14 \\ & y_1, y_2, y_3 \geq 0 \end{aligned} \quad \rightarrow$$

$$\begin{aligned} & \min \quad 1200y_1 + 1000y_2 + 700y_3 \\ \text{s.t.} \quad & 2y_1 + y_2 + y_3 \geq 24 \\ & 3y_1 + 4y_2 + 2y_3 \geq 14 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

Dual LP

	x^T	
y	A	$\geq b$
	$\leq c^T$	

In matrix form:

If LP is in normal form

➤ Symmetric dual

(P)

D	x_1	x_2	\dots	x_n	
y_1	a_{11}	a_{12}	\dots	a_{1n}	$\geq b_1$
y_2	a_{21}	a_{22}	\dots	a_{2n}	$\geq b_2$
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
y_m	a_{m1}	a_{m2}	\dots	a_{mn}	$\geq b_m$
	$\leq c_1$	$\leq c_2$	\dots	$\leq c_n$	

Otherwise

➤ Asymmetric dual:

- Bring primal (P) to normal form (P')
- Construct dual (D) for (P')

Example 2

➤ Primal LP:

$$\min z = 3x_1 + x_2$$

$$\text{s.t. } 2x_1 + 5x_2 \geq 7$$

$$x_1 + 4x_2 \geq 2$$

$$x_1, x_2 \geq 0$$

→ NORMAL
FORM

➤ Dual LP:

$$\max w = 7y_1 + 2y_2$$

s.t.

$$2y_1 + y_2 \leq 3$$

$$5y_1 + 4y_2 \leq 1$$

$$y_1, y_2 \geq 0$$

Example 3

$$\min z = 4x_1 + 12x_2 + x_3$$

$$\text{s.t. } -x_1 - 4x_2 + x_3 \leq 1$$

$$2x_1 + 2x_2 + x_3 = 2$$

$$x_1, x_2 \geq 0; \quad x_3 \text{ - URS}$$

not in
normal
form

(P)

$$\downarrow \quad \text{let } x_3 = x_3' - x_3''$$

$$\min z = 4x_1 + 12x_2 + x_3' - x_3''$$

$$\text{s.t. } x_1 + 4x_2 - x_3' + x_3'' \geq -1$$

$$2x_1 + 2x_2 + x_3' - x_3'' \geq 2$$

$$-2x_1 - 2x_2 - x_3' + x_3'' \geq -2$$

$$x_1, x_2, x_3', x_3'' \geq 0$$

(P)

in normal
form

$$\max \quad -y_1 + 2y_2 - 2y_3$$

$$\text{s.t. } y_1 + 2y_2 - 2y_3 \leq 4$$

$$4y_1 + 2y_2 - 2y_3 \leq 12$$

$$-y_1 + y_2 - y_3 \leq 1$$

$$y_1 - y_2 + y_3 \leq -1$$

$$y_1, y_2, y_3 \geq 0$$

Dual
in normal
form

$$\text{max } -y_1 + 2y_2 - 2y_3$$

$$\text{s.t. } y_1 + 2y_2 - 2y_3 \leq 4$$

$$4y_1 + 2y_2 - 2y_3 \leq 12$$

$$-y_1 + y_2 - y_3 \leq 1$$

$$y_1 - y_2 + y_3 \leq -1$$

$$y_1, y_2, y_3 \geq 0$$

$$\left. \begin{array}{l} -y_1 + y_2 - y_3 \leq 1 \\ -y_1 + y_2 - y_3 \geq 1 \end{array} \right\} \rightarrow \frac{-y_1 + y_2 - y_3 = 1}{-y_1 + y_2 - y_3 = 1}$$

$$\text{let } y_2 - y_3 = \tilde{y}_2 \rightarrow \tilde{y}_2 - \text{URS}$$

$$\tilde{y}_1 = -y_1 \rightarrow \tilde{y}_1 \leq 0$$

$$\text{max } \tilde{y}_1 + 2\tilde{y}_2$$

$$\text{s.t. } -\tilde{y}_1 + 2\tilde{y}_2 \leq 4$$

$$-4\tilde{y}_1 + 2\tilde{y}_2 \leq 12$$

$$\tilde{y}_1 + \tilde{y}_2 = 1$$

$$\tilde{y}_1 \leq 0, \tilde{y}_2 - \text{URS}$$

$$\begin{array}{l}
 \min z = 4x_1 + 12x_2 + x_3 \\
 \text{s.t.} \quad -x_1 - 4x_2 + x_3 \leq 1 \\
 \quad \quad 2x_1 + 2x_2 + x_3 = 2 \\
 \quad \quad x_1, x_2 \geq 0; x_3 \text{ - URS}
 \end{array}
 \left. \vphantom{\begin{array}{l} \min z \\ \text{s.t.} \end{array}} \right\} \begin{array}{l} \text{not in} \\ (x \text{ - } 1) \\ \text{normal} \\ \rightarrow \leq \geq \\ \text{form} \end{array} \quad (P)$$

$$\begin{array}{l}
 \max z \quad \bar{y} + 2\tilde{y} \\
 \text{s.t.} \quad -\bar{y} + 2\tilde{y} \leq 4 \\
 \quad \quad -4\bar{y} + 2\tilde{y} \leq 12 \\
 \quad \quad \bar{y} + \tilde{y} = 1 \\
 \quad \quad \bar{y} \leq 0, \tilde{y} \text{ - URS}
 \end{array}
 \left. \vphantom{\begin{array}{l} \max z \\ \text{s.t.} \end{array}} \right\} \begin{array}{l} \text{dual} \\ \text{not} \\ \text{in} \\ \text{normal} \\ \text{form} \end{array}$$

Primal

1st primal constr $(x \text{ - } 1) \rightarrow \bar{y} \leq 0$
 2nd primal constr $= \rightarrow \tilde{y} \text{ - URS}$

Dual

x_3 is URS \rightarrow 3rd dual constraint
 $\bar{y} \leq 0 \rightarrow$ reversed " = " 1st dual constraint

Example 4*

➤ Primal LP:

$$\max z = 6x_1 + x_2 + x_3$$

s.t.

$$4x_1 + 3x_2 - 2x_3 = 1$$
$$6x_1 - 2x_2 + 9x_3 \geq 9$$
$$2x_1 + 3x_2 + 8x_3 \leq 5$$
$$x_1 \geq 0, x_2 \leq 0, x_3 \text{ - urs}$$

➤ Dual LP:

* from Linear and Non-Linear Programming by S.G.Nash and A.Sofer

Example 4*

* from Linear and Non-Linear Programming by S.G.Nash and A.Sofer

Dual LP

	x_1	x_2	\dots	x_n	
y_1	a_{11}	a_{12}	\dots	a_{1n}	$\geq b_1$
y_2	a_{21}	a_{22}	\dots	a_{2n}	$\geq b_2$
y_i					$\leq b_i$
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
y_j					$= b_j$
y_m	a_{m1}	a_{m2}	\dots	a_{mn}	$\geq b_m$
	$\leq c_1$	$\leq c_2$	\dots	$\leq c_n$	

➤ Asymmetric dual:

apply the following rules:

primal/dual constraint		dual/primal variable
consistent with normal form	\iff	variable ≥ 0
reversed with normal form	\iff	variable ≤ 0
equality constraint	\iff	variable urs

Example 4*

➤ Primal LP:

$$\max z = 6x_1 + x_2 + x_3$$

s.t.

$$4x_1 + 3x_2 - 2x_3 = 1$$
$$6x_1 - 2x_2 + 9x_3 \geq 9$$
$$2x_1 + 3x_2 + 8x_3 \leq 5$$
$$x_1 \geq 0, x_2 \leq 0, x_3 \text{ - urs}$$

➤ Dual LP:

* from Linear and Non-Linear Programming by S.G.Nash and A.Sofer

Duality theory

Lemma 1 The dual of the dual is the primal.

Example:

➤ Primal LP:

$$\min z = 3x_1 + x_2$$

$$\text{s.t. } 2x_1 + 5x_2 \geq 7$$

$$x_1 + 4x_2 \geq 2$$

$$x_1, x_2 \geq 0$$

➤ Dual LP:

HW

Show that dual of the dual is the primal problem

General case:

$$\begin{aligned} \text{(P)} \quad & \min \quad c^T x \\ & \text{s.t.} \quad Ax \geq b \\ & \quad \quad x \geq 0 \end{aligned}$$

→

$$\begin{aligned} \text{(D)} \quad & \max \quad b^T y \\ & A^T y \leq c \\ & \quad \quad y \geq 0 \end{aligned}$$

Dual of the dual:

1. (D) in min form :

$$\min -b^T y$$

s.t.

$$-A^T y \geq -c$$

$$y \geq 0$$

2. Dual of the above

$$\max -c^T z$$

s.t.

$$(-A^T)^T z \leq -b \quad \rightarrow \quad -Az \leq -b$$

$$z \geq 0$$

3. Presenting 2. in min form:

$$\min c^T z$$

s.t.

$$Az \geq b$$

$$z \geq 0$$

\leftrightarrow (P)

Weak duality theorem

Assume (P) is in min form

For a feasible solution x to the primal LP (P) and a feasible solution y to the dual LP (D),

$$c^T x \geq b^T y$$

In other words, $z \geq w$.

Example

➤ Primal LP:

$$\min z = 3x_1 + x_2$$

$$\text{s.t. } 2x_1 + 5x_2 \geq 7$$

$$x_1 + 4x_2 \geq 2$$

$$x_1, x_2 \geq 0$$

➤ Dual LP: $\max w = 7y_1 + 2y_2$

$$\text{s.t. } 2y_1 + y_2 \leq 3$$

$$5y_1 + 4y_2 \leq 1$$

$$y_1, y_2 \geq 0$$

Compare values of Z and W for any two feasible solutions for primal and dual problems

$$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow \text{Feasible}$$
$$z(1, 1) = 4$$

$$y^T = (0.1, 0.1) \rightarrow \text{Feasible}$$

$$w(0.1, 0.1) = 0.9$$

$$z(1, 1) = 4 > 0.9 = w(0.1, 0.1)$$

$$z^* \geq 0.9 \downarrow \text{-LB for (P)}$$

$$w^* \leq 4 \text{ - UP for (D)}$$

Weak duality theorem

Let $x_1 = 0$

$$(P) \begin{cases} 5x_2 \geq 7 \\ 4x_2 \geq 2 \end{cases} \rightarrow \begin{cases} x_2 \geq \frac{7}{5} \\ x_2 \geq \frac{1}{2} \end{cases} \rightarrow \min x_2 = \frac{7}{5}$$

$$x = \begin{pmatrix} 0 \\ \frac{7}{5} \end{pmatrix};$$

$$z = \frac{7}{5}$$

$$(D) \begin{cases} 2y_1 \leq 3 \\ 5y_1 \leq 1 \end{cases} \rightarrow \begin{cases} y_1 \leq \frac{3}{2} \\ y_1 \leq \frac{1}{5} \end{cases} \rightarrow \max y_1 = \frac{1}{5}$$

$$y = \begin{pmatrix} \frac{1}{5} \\ 0 \end{pmatrix}$$

$$\rightarrow w = \frac{7}{5}$$

$$z = \frac{7}{5} \geq \frac{7}{5} = w$$

$z = LB \rightarrow z$ is optimal
 $w = UB \rightarrow w$ is optimal

Weak duality theorem – optimality gap

➤ Primal LP:

$$\min z = 5x_1 + 2x_2$$

$$\text{s.t. } x_1 - x_2 \geq 3$$

$$2x_1 + 3x_2 \geq 5$$

$$x_1, x_2 \geq 0$$

$$x = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad z = 15$$

➤ Dual LP:

$$\max w = 3y_1 + 5y_2$$

$$\text{s.t. } y_1 + 2y_2 \leq 5$$

$$-y_1 + 3y_2 \leq 2$$

$$y_1, y_2 \geq 0.$$

$$y = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad w = 3$$

➤ How can we estimate the optimality gap based on the weak duality theorem?

$$\text{OG} : \frac{15 - 3}{3} = 400\% \rightarrow \text{too wide}$$

$$y = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \rightarrow w = 12$$

$$\frac{15 - 12}{3} = 100\%$$

$$y = \begin{pmatrix} 4.5 \\ 0 \end{pmatrix} \rightarrow w = 13.5 \rightarrow \frac{15 - 13.5}{3} = 50\%$$

$$y = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \rightarrow w = 15 \rightarrow \frac{15 - 15}{3} = 0\% \rightarrow z^* = w^* = 15$$

optimality gap:
for min

$$\frac{\text{Best solution} - \text{LB}}{\text{LB}}$$

$$\frac{7/5 - 7/5}{7/5} = 0$$

for max

$$\frac{\text{UB} - \text{Best Feas.}}{\text{UB}}$$

Weak duality theorem

Corollary 3

- 1) If the primal LP is unbounded, then the dual LP is infeasible.
- 2) If the dual LP is unbounded, then the primal LP is infeasible.

$$1. \quad z = w$$

Example (P)

$$\max z = 2x_1 + 3x_2$$

$$\text{s.t.} \quad x_1 - x_2 \leq 1$$

$$x_1 - 2x_2 \leq 2$$

$$x_1, x_2 \geq 0$$

$$\rightarrow \begin{array}{l} + s_1 \\ + s_2 \end{array}$$

$$x_{B_0} = (s_1, s_2)$$

	x_1	x_2	s_1	s_2	
z	-2	-3	0	0	0
s_1	1	-1	0	1	1
s_2	1	-2	0	1	2

as $\hat{a}_{i2} < 0 \rightarrow$ unbounded

Corollary - example

(D)

$$\min w = y_1 + 2y_2$$

s.t.

$$y_1 + y_2 \geq 2$$

$$-y_1 - 2y_2 \geq 3$$

$$y_1, y_2 \geq 0.$$

→ we will try to solve it
with dual simplex

(P)

$$\max z = 2x_1 + 3x_2$$

$$\text{s.t. } x_1 - x_2 \leq 1$$

$$x_1 - 2x_2 \leq 2$$

$$x_1, x_2 \geq 0$$

Strong duality theorem

Let x be a feasible solution for primal LP (P) and y be a feasible solution of the corresponding dual LP (D). Then

$$c^T x = b^T y$$

if and only if x is an optimal solution for (P) and y is an optimal solution for (D).

(We need the dual of the LP in the standard form)

Corollary

If an optimal solution to the dual LP (D) is obtained, an optimal solution to its primal LP (P) can be readily obtained and both optimal objective values are equal.

Strong duality theorem

Example : solve primal /dual of Example 2, apply the strong duality theorem to the corresponding dual/primal problem to find its optimal solution.

Example

➤ Primal LP:

$$\min z = 3x_1 + x_2$$

$$\text{s.t. } 2x_1 + 5x_2 \geq 7$$

$$x_1 + 4x_2 \geq 2$$

$$x_1, x_2 \geq 0$$

➤ Dual LP: $\max w = 7y_1 + 2y_2$
s.t. $2y_1 + y_2 \leq 3$
 $5y_1 + 4y_2 \leq 1$
 $y_1, y_2 \geq 0$

Solve dual:

$$\max w = 7y_1 + 2y_2$$

$$\text{s.t. } 2y_1 + y_2 + s_1 = 3$$

$$5y_1 + 4y_2 + s_2 = 1$$

$$y_1, y_2, s_1, s_2 \geq 0$$

$$x_{B_1} = (s_1, s_2)$$

$$\rightarrow w - 7y_1 - 2y_2 = 0$$

$$\text{or } \hat{c}_N^T = c_B^T B^{-1} N - c_N^T =$$
$$= (0, 0) - c_N^T =$$

$$= (-7, -2)$$

$$z_0 = c_B^T B^{-1} b = 0$$

Strong duality theorem

	y_1	y_2	s_1	s_2	RHS
w	-7	-2	0	0	0
s_1	2	1	1	0	3
s_2	5	4	0	1	1
w	0	$18/5$	0	$7/5$	$7/5$
s_1	0	$-3/5$	1	$-2/5$	$13/5$
y_1	1	$4/5$	0	$1/5$	$1/5$

$3/2$
 $1/5$
 $+ 7R_1$

optimal, as $\hat{C}_x > 0$.

$w^* = 7/5$
 $y_1^* = 1/5$
 $y_2^* = 0$

By strong duality theorem,

$$z^* = w^* = \frac{7}{5} \rightarrow$$

$$z^* = 3x_1 + x_2 = \frac{7}{5} \rightarrow \text{need another eq-n to find } x_1^* \quad x_2^*$$



complementary slackness theorem

as $y_1^* = \frac{1}{5} \neq 0 \rightarrow 1^{\text{st}}$ primary constraint is active:

$$\begin{cases} 2x_1 + 5x_2 = 7 & (1) \div 5 \\ 3x_1 + x_2 = \frac{7}{5} & (2) \end{cases}$$

$$\frac{2}{5}x_1 + x_2 - 3x_1 - x_2 = 0$$

$$x_1 \left(\frac{2}{5} - 3 \right) = 0$$

$$x_1 = 0 \rightarrow x_2 = \frac{7}{5}$$

$$x^* = \begin{pmatrix} 0 \\ \frac{7}{5} \end{pmatrix}$$

Complementary slackness theorem

Theorem 6

- Let x be a feasible solution to the primal LP (P) and y be a feasible solution to the dual LP (D). Both solutions x and y are optimal to the primal (P) and dual (D), respectively, if and only if they satisfy

$$(c^T - y^T A)x = 0 \quad \text{and} \quad y^T (b - Ax) = 0 \quad (2)$$

THAT IS:

for each i^{th} dual
 $(c_i - y^T A_i)x_i = 0$

and

for each j^{th} primal
 $y_j (b_j - A_j x) = 0.$

constraint:

if in opt x , $x_i \neq 0$

$$c_i - y^T A_i = 0$$

(active constraint)

constraint:

if in opt y , $y_j \neq 0$

↓

$$b_j - A_j x = 0$$

(active constraint)

binding

Complementary slackness theorem

➤ Example 7 : consider the LP

$$\min z = 2x_1 + 2x_2$$

$$\begin{aligned} \text{s.t.} \quad & 2x_1 + x_2 \geq 6 \\ & x_1 + 2x_2 \geq 6 \\ & x_1, x_2 \geq 0 \end{aligned} \quad (P) \quad \rightarrow \quad (D)$$

$$\begin{aligned} \max w &= 6y_1 + 6y_2 \\ \text{s.t.} \quad & 2y_1 + y_2 \leq 2 \\ & y_1 + 2y_2 \leq 2 \\ & y_1, y_2 \geq 0 \end{aligned}$$

Solve the dual problem and, with the Complementary slackness Theorem, obtain the solution for the primal problem.

(D) → standard form:

$$\begin{aligned} \max w &= 6y_1 + 6y_2 \\ \text{s.t.} \quad & 2y_1 + y_2 + s_1 = 2 \\ & y_1 + 2y_2 + s_2 = 2 \\ & s_1, s_2, y_1, y_2 \geq 0 \end{aligned}$$

$$x_{B_0} = (s_1, s_2)$$

	y_1	y_2	s_1	s_2	RHS
w	-6	-6	0	0	0
s_1	2	1	1	0	2
s_2	1	2	0	1	2

Complementary slackness theorem

	y_1	y_2	s_1	s_2	RHS
w	-6	-6	0	0	0
s_1	2	1	1	0	2
s_2	1	2	0	1	2
w	0	-3	3	0	6
y_1	1	1/2	1/2	0	1
s_2	0	3/2	-1/2	1	1
w	0	0	2	2	8
y_1	1	0	2/3	-1/3	2/3
y_2	0	1	-1/3	2/3	2/3

$2 \div 2$
 $2 \div 1$
 $+ 6 R_1'$
 $1 \div \frac{1}{2}$
 $1 \div \frac{3}{2}$
 $R_0' = R_0 + 3 R_2'$
 $R_1' = R_1 - \frac{1}{2} R_2'$
 $R_2' = R_2 \times \frac{2}{3}$

* from Linear and Non-Linear Programming by S.G.Nash and A.Sofer



opt. as $\hat{C}_N > 0 \rightarrow$

$$W^* = 8 \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \rightarrow z^* = 8$$

$y_1^* = \frac{2}{3} \neq 0$
 $y_2^* = \frac{2}{3} \neq 0$ } \rightarrow both constraints in (P) are active

$$z^* = 8 = 2x_1 + 2x_2 \rightarrow \begin{cases} x_1 + x_2 = 4 & \textcircled{1} \\ 2x_1 + x_2 = 6 & \textcircled{2} \end{cases}$$

1st constr.

$$\begin{array}{r} x_1 + x_2 = 4 \\ -2x_1 - x_2 = -6 \\ \hline -x_1 + 2x_2 = -2 \end{array}$$

$x_1^* = 2; x_2^* = 2$

$$x^* = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

Summary

Dual

	Finite optimum	unbounded	infeasible
Primal	Finite optimum	—	—
	unbounded	—	✓
	infeasible	✓	✓

↑
provide an example 😊

Dual Simplex method - summary

- Primal Simplex – finding series of solutions satisfying *primal* feasibility condition

$$x_b = B^{-1}b \geq 0$$

till finds a solution satisfying *primal* optimality condition:

$$\widehat{c}_N^T \leq 0 \quad (\text{if primal is a min problem}),$$

- Dual Simplex – finding series of solutions satisfying dual feasibility condition till finds a solution satisfying *dual* optimality condition:

$$\widehat{b}_N^T \geq 0 \quad (\text{dual is a max problem then})$$

- Tableau: working out in reverse order – finding the leaving variable first.

- Use for

- 1) generating initial bfs and
- 2) recovering feasibility

$$(P) \quad \begin{aligned} \min \quad & c^T x \\ Ax \geq & b \\ x \geq & 0 \end{aligned}$$

$$(D) \quad \begin{aligned} \max \quad & b^T y \\ A^T y \leq & c \\ y \geq & 0. \end{aligned}$$

↓

1. Assume for x_B
 $x_B = B^{-1}b$

2. Assume that $y = (B^{-1})^T c_B$ is feasible for (D):

$$A^T y \leq c \rightarrow A^T (B^{-1})^T c_B \leq c$$

$$\downarrow$$

$$c_B^T B^{-1} A - c^T \leq 0^T$$

$$\downarrow$$

$$\hat{c} \leq 0$$

$$\downarrow$$

$$\hat{c}_N \leq 0 \quad \hat{b} \geq 0$$

↓

if $x_B = B^{-1}b \geq 0 \rightarrow$
 x_B is optimal

(D) Feasibility \rightarrow (P) optimality

(P) Feasibility \rightarrow (D) optimality

Dual Simplex method

- Step 1 (feasibility test) If $\hat{b} = B^{-1} b \geq 0$, then STOP – current primal bfs is feasible;

Otherwise, select the variable $(x_B)_s$ whose rhs \hat{b}_s is the most negative among components of \hat{b} . This component of the primal basis will be leaving

- Step 2 (entering) If all entries \hat{a}_{sj} in the row corresponding to the leaving variable $(x_B)_s$ are non-negative ($\hat{a}_{sj} \geq 0$, for any $j = 1, 2, \dots, n$), then STOP – the considered LP (the primal) is infeasible. Otherwise, find

$$t = \arg \min_j \left\{ \left| \frac{\hat{c}_j}{\hat{a}_{sj}} \right| : \hat{a}_{sj} < 0 \right\}$$

- Step 3 (pivoting) Update the tableau by pivoting on \hat{a}_{sj} , i.e. perform EROs on the tableau to get a 1 in the pivot position, and 0s above and below it. GO TO Step 1.

Generating initial bfs

➤ Example 9

$$\max z = 4x_1 + 5x_2$$

$$\text{s.t. } 2x_1 + 3x_2 \leq 6$$

$$3x_1 + x_2 \geq 3$$

$$x_1, x_2 \geq 0$$

St. Form:

$$\max z = 4x_1 + 5x_2$$

$$\text{s.t. } 2x_1 + 3x_2 + s_1 = 6$$

$$3x_1 + x_2 - e_2 = 3 \quad \times(-1)$$

$$-3x_1 - x_2 + e_2 = -3$$

$$x_1, x_2, s_1, e_2 \geq 0$$

$$x_B = (s_1, e_2)$$

	x_1	x_2	s_1	e_2	RHS
z	-4	-5	0	0	0
s_1	2	3	1	0	6
e_2	-3	-1	0	1	-3

NOT feasible

Ratio Test:

$$\left| \frac{-4}{-3} \right| < \left| \frac{-5}{-1} \right|$$

x_1 enters

$-3 < 0$: e_2 must leave

Feas, not optimal

$$R_0' = R_0 + 4R_2'$$

$$R_1' = R_1 - 2R_2'$$

leaving

(max problem!) $R_2' = R_2 / (-3)$

z	0	$-\frac{11}{3}$	0	$-\frac{4}{3}$	4
s_1	0	$\frac{7}{3}$	1	$\frac{2}{3}$	4
x_1	1	$\frac{1}{3}$	0	$-\frac{1}{3}$	1

UTS

z	0	0	$\frac{11}{7}$	$-\frac{2}{7}$	$\frac{72}{7}$
x_2	0	1	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{12}{7}$
x_1	1	0	$-\frac{1}{7}$	$-\frac{3}{7}$	$\frac{3}{7}$
z	0	1	2	0	12
e_2	0	$\frac{7}{2}$	$\frac{3}{2}$	1	6
x_1	1	$\frac{3}{2}$	$\frac{1}{2}$	0	3

Optimal, $\hat{c}_N > 0$

↓

$$z^* = 12 \quad x = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

Ratio:
 $4 \times \frac{3}{7} < 3$

$$R_1' = R_1 \times \frac{3}{7}$$

$$R_2' = R_2 - \frac{1}{3} R_1'$$

$$R_0' = R_0 + \frac{11}{3} R_1'$$

$$\times \frac{7}{2}$$

$$R_2' = R_2 + \frac{3}{7} R_1'$$

$$R_0' = R_0 + R_1'$$

Recovering the Feasibility

Example 10

$$\max z = 0.1x_1 + 0.15x_2$$

$$\text{s.t.} \quad x_1 + x_2 \leq 100000 \quad + S_1$$

$$-\frac{3}{4}x_1 + \frac{1}{4}x_2 \leq 0 \quad + S_2$$

$$-\frac{3}{2}x_1 + x_2 \leq 0 \quad + S_3$$

$$x_1, x_2 \geq 0$$

$$x_{B_0} = (s_1 \ s_2 \ s_3)$$

	x_1	x_2	s_1	s_2	s_3	RHS
z	-0.1	-0.15	0	0	0	0
s_1	1	1	1	0	0	100000
s_2	$-\frac{3}{4}$	$\frac{1}{4}$	0	1	0	0
s_3	$-\frac{3}{2}$	1	0	0	1	0

	x_1	x_2	s_1	s_2	s_3	RHS
z	-0.1	-0.15	0	0	0	0
s_1	1	1	1	0	0	10000
s_2	$-\frac{3}{4}$	$\frac{1}{4}$	0	1	0	0
s_3	$-\frac{3}{2}$	1	0	0	1	0
z	0.05	0	0.15	0	0	15000
x_2	1	1	1	0	0	10000
s_2	-1	0	$-\frac{1}{4}$	1	0	-25000 < 0
s_3	$-\frac{5}{2}$	0	-1	0	1	-100000 < 0
z	0	0	0.13	0	0.02	13000
x_2	0	1	0.6	0	0.4	60,000
s_2	0	0	0.15	1	-0.4	15,000
x_1	1	0	$\frac{2}{5}$	0	$-\frac{2}{5}$	40,000

s_1 is leaving as choosing s_2 or s_3 will not change z

$R'_0 = R_0 + 0.15R_1$
 $R'_2 = R_2 - \frac{1}{4}R_1$
 $R'_3 = R_3 - R_1$

↑ optimal, not feasible

s_3 is leaving as $-100 < -25$
 $R'_3 = R_3 \times (-\frac{2}{5})$

Optimal as $\hat{C}_N > 0$

$z^* = 13000$
 $x_1^* = 40,000$
 $x_2^* = 60,000$

Ratio:
 $\left| \frac{0.05}{-2.5} \right| < \left| \frac{0.15}{-1} \right|$
 x_1 enters
 $R'_2 = R_2 + R_3$
 $R'_1 = R_1 - R_3$
 $R'_0 = R_0 - 0.05R_3$

Dual Simplex method – examples*

Use dual Simplex method to solve:

$$\min z = 5x_1 + 4x_2$$

$$\text{s.t.} \quad 4x_1 + 3x_2 \geq 10$$

$$3x_1 - 5x_2 \geq 12$$

$$x_1, x_2 \geq 0$$

* from Linear and Non-Linear Programming by S.G.Nash and A.Sofer

Dual Simplex method – examples*

Use dual Simplex method to solve:

$$\max z = -2x_1 - 7x_2 - 6x_3 - 5x_4$$

$$\text{s.t.} \quad 2x_1 - 3x_2 - 5x_3 - 4x_4 \geq 20$$

$$7x_1 + 2x_2 + 6x_3 - 2x_4 \geq 35$$

$$4x_1 + 5x_2 - 3x_3 - 2x_4 \geq 15$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$x_B = (e_1, e_2, e_3)$$

$$\max z = -2x_1 - 7x_2 - 6x_3 - 5x_4$$

s. +.

$$\begin{cases} 2x_1 - 3x_2 - 5x_3 - 4x_4 - e_1 = 20 \\ 7x_1 + 2x_2 + 6x_3 - 2x_4 - e_2 = 35 \\ 4x_1 + 5x_2 - 3x_3 - 2x_4 - e_3 = 15 \\ x_1, x_2, x_3, x_4 \geq 0 \\ e_1, e_2, e_3 \geq 0 \end{cases}$$

Bring to canonical form
by $\times (-1)$

* from Linear and Non-Linear Programming by S.G.Nash and A.Sofer

	x_1	x_2	x_3	x_4	e_1	e_2	e_3	RHS
z	2	7	6	5	0	0	0	0
e_1	-2	3	5	4	1	0	0	-20 < 0
e_2	-7	-2	-6	2	0	1	0	-35 < 0 ^{most neg-ve}
e_3	-4	-5	3	2	0	0	1	-15 < 0

①

z	0	$45/7$	$30/7$	$39/7$	0	$2/7$	0	-10
e_1	0	$25/7$	$47/7$	$24/7$	1	$-2/7$	0	-10 < 0 → must leave
x_1	1	$2/7$	$6/7$	$-2/7$	0	$-1/7$	0	5
e_3	0	$-27/7$	$45/7$	$6/7$	0	$-4/7$	1	5

②

z	0	10	11	9	1	0	0	-20 $R_0' = R_0 + R_1$
e_2	0	$-\frac{25}{2}$	$-\frac{47}{2}$	-12	$-\frac{1}{2}$	1	0	35 $R_1' = R_1 \times (-\frac{7}{2})$
x_1	1	$-\frac{21}{14}$	$-\frac{35}{14}$	-2	$-\frac{1}{2}$	0	0	10 $R_2' = R_2 - \frac{1}{2}R_1$
e_3	0	-11	-7	-6	-2	0	1	25 $R_3' = R_3 - 2R_1$

③

③ Feasible and optimal as $C_N > 0$

$$z^* = -20$$

$$x^* = \begin{pmatrix} 10 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

① Ratio: $\left| \frac{2}{-7} \right| \left| \frac{7}{-2} \right| \left| \frac{6}{-6} \right|$

$R_0' = R_0 - 2R_2'$ min → x_1 enters

$R_1' = R_1 + 2R_2'$

$R_2' = \frac{R_2}{-7}$

$R_3' = R_3 + 4R_2'$

② Optimal, but NOT feasible
Ratio: as only one element of 1st row is negative, e_2 enters

