Introduction to Optimisation:

Simplex Method

Lecture 3

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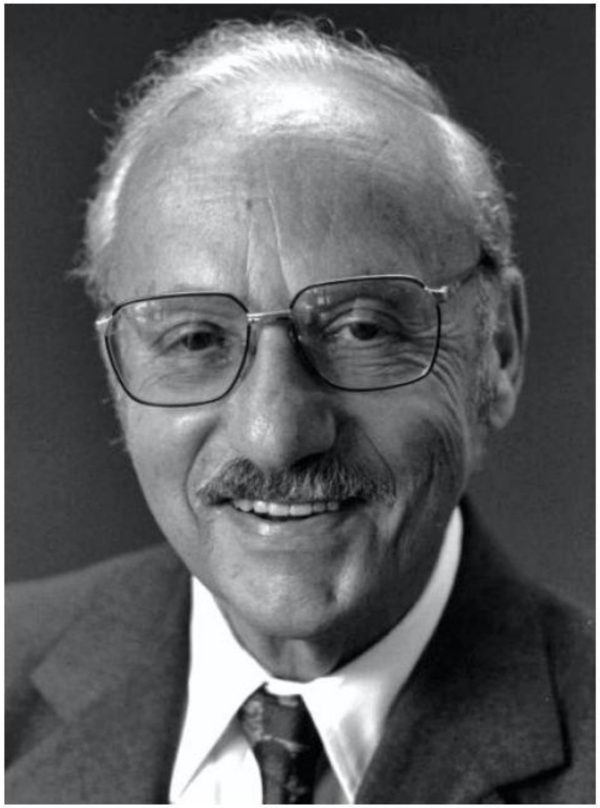
Simplex method



George Dantzig (1914 - 2005) invented the simplex method in 1947.

In 1954 Dantzig together with Orchard Hays developed the revised simplex algorithm.

Simplex method

- Simplex demonstration
  - https://youtu.be/k9em_7B6298?si=XVchpO-RjaPbMgUf
  - https://youtu.be/k9em_7B6298?si=dgRTTLFmEuFTiacS
- Simplex method performs an efficient search of the extreme points (i.e. bfs) of the feasible region. The method usually starts from the bfs where all original decision variables are zeros.
- Then it "greedily" (in the sense that the objective function value is getting improved) moves from one extreme point (i.e. bfs) of the feasible region to an adjacent bfs by changing one basic variable at a time.
- In the searching/moving procedure, the *ratio test* ensures that the basic solution in each iteration remains feasible (i.e. satisfies all constraints). The method ceases when no further improvement in the value of the objective function
- For any LP with m constraints, two bfs are said to be *"adjacent"* if their bases have $m - 1$ basic variables in common.

$n = 5$

$m = 3$

Example

- Solve:

$$\begin{aligned} \min z = & -x_1 - 2x_2 \\ s.t. \quad & -2x_1 + x_2 \le 2 \\ & -x_1 + 2x_2 \le 7 \\ & x_1 \le 3 \\ & x_1,\ x_2 \ge 0 \end{aligned}$$

$$A = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & s_3 \end{array} \\ \begin{pmatrix} -2 & 1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} \\ 3\times 5 \quad \underbrace{\begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \end{array}}_{\text{columns of } A} \end{array}$$

- Standard form:

$$\begin{aligned} \min z = & -x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 \\ s.t. \quad & -2x_1 + x_2 + s_1 = 2 \\ & -x_1 + 2x_2 + s_2 = 7 \\ & x_1 + s_3 = 3 \\ & x_1,\ x_2,\ s_1\ s_2\ s_3 \ge 0 \end{aligned}$$

$|x_B| = 3$

$|x_N| = 2$

- Each of the constraints has a <u>unique</u> variable.

Initial basis

- $x_B = (\ s_1\ s_2\ s_3\ );\quad x_N = (\ x_1\ x_2\ );$

- Hence **bfs** $x = (\ \underset{x_1}{0}\ \ \underset{x_2}{0}\ \ \underset{s_1}{2}\ \ \underset{s_2}{7}\ \ \underset{s_3}{3}\ )$ and the corresponding value of $z = 0$

is current BFS optimal ?

$-2x_1 + x_2 + s_1 = 2$
$-x_1 + 2x_2 + s_2 = 7$
$x_1 + s_3 = 3$

Example

- ➢ Finding an *adjacent* bfs to improve z:

(What an adjacent bfs?)

1. Express every component of x_B in terms of x_N :

$s_1 = 2 + 2x_1 - x_2$
$s_2 = 7 + x_1 - 2x_2$
$s_3 = 3 - x_1$

2. Express z in equality form: $z = -x_1 - 2x_2$

$z + x_1 + 2x_2 = 0$

min problem

3. All OF coefficients for x_N are positive/negative, chose the one with most positive/negative coefficient to enter basis:

To improve z chose x_2, and let $x_1 = 0.$

3. To determine the limits of increase for x_2 solve:

$s_1 = 2 - x_2 \geq 0 \rightarrow x_2 \leq 2 \rightarrow x_2 = 2, \rightarrow s_1 = 0$ leaving
$s_2 = 7 - 2x_2 \geq 0 \qquad x_2 \leq 7/2$
$s_3 = 3$

Hence x_2 is entering the basis and s is leaving the basis

Example

$z + 4 = 0$

$z + \underset{0}{\underset{\parallel}{x_1}} + 2\underset{2}{\underset{\parallel}{x_2}} = 0$

- New values for components:

New bfs $(x_1, x_2, \underline{s_1\ s_2\ s_3}) = (0\ \ 2\ \ 0\ \ 3\ \ 3)$ and $z = -4$

- New $x_B = (x_2\ s_2\ s_3)$; $x_N = (x_1\ \ s_1\ \ )$

$x_2 = 2 + 2x_1 - s_1$

$s_2 = 7 + x_1 - 2(2 + 2x_1 - s_1) = 3 - 3x_1 + 2s_1$

$s_3 = 3 - x_1$

1. Express every component of x

- $s_1 = 2 + 2x_1 - x_2$
- $s_2 = 7 + x_1 - 2x_2$
- $s_3 = 3 - x_1$

- Express z and every component of x_B in terms of x_N:

$$z + x_1 + \underbrace{4 + 4x_1 - 2s_1}_{2x_2} = 0$$

$$z + 5x_1 - 2s_1 + 4 = 0.$$

- Can we approve z further? Some coefficients for x_N are positive

yes

Example

- To improve z chose __x_1__, and let __s_1__ $= 0$.

To determine the limits of increase of __x_1__ solve:

$x_2 = 2 + 2x_1 \geq 0 \rightarrow x_1 \geq -1$ unlimited Increa

$s_2 = 3 - 3x_1 \geq 0 \rightarrow x_1 \leq 1 \rightarrow x_1 = 1$, s_2 is leaving basis

$s_3 = 3 - x_1 \geq 0 \rightarrow x_1 \leq 3$

- New values for components:

$z + 5x_1 - 2s_1 = -4$

- New bfs $(x_1, x_2, \overset{s_1}{\cancel{x_3}}, \overset{s_2}{\cancel{x_4}}, \overset{s_3}{\cancel{x_5}}) = (1 \quad 4 \quad 0 \quad 0 \quad 2)$ and $z = -9$
- New $x_B = (x_1 \quad x_2 \quad s_3)$; $x_N = (s_1 \quad s_2)$;

….and so on…

Simplex Method in a general form

- $\min\ z = c^T x$

  s.t. $Ax = b,$

  $x \geq 0$

where

$x = (x_1 \ldots\ldots x_n)^T$

$c = (c_1 \ldots\ldots c_n)^T$

$b = (b_1 \ldots b_m)^T$

$$A = [A_1, A_2, \ldots, A_n] = \begin{array}{cccc} x_1 & x_2 & & x_n \\ \left(\begin{matrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{matrix}\right. & \begin{matrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{matrix} & & \left.\begin{matrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{matrix}\right) \\ A_1 & A_2 & \ldots\ldots & A_n \end{array}$$

$rank\ A = m$

$n > m$

Simplex Method in a general form

- Select m variables with linear independent columns in A:

x_B - basis; $B(m \times m)$ - basic matrix;

order is important :

if $x_B^T = (x_5\ x_2\ x_1) \to c_B^T = (c_5\ c_2\ c_1)$

x_N - non-basis; $N(m \times (n-m))$ -non-basic matrix.

Then

$x^T = [\ x_B^T \mid x_N^T\ ]$

$c^T = [\ c_B^T \mid c_N^T\ ]$

$A = [\ B \mid N\ ]$

- Re-write the LP in a *partition form*:

$$\min z = c^T x = (c_B^T \mid c_N^T)\begin{pmatrix} x_B \\ x_N \end{pmatrix} = c_B^T x_B + c_N^T x_N$$

$$\text{s.t.} \quad Ax = b$$

$$[B \mid N]\begin{bmatrix} x_B \\ x_N \end{bmatrix} = b$$

$$B x_B + N x_N = b$$

$$x_B \geq 0 \quad x_N \geq 0$$

Simplex Method in a general form

- Step 1: current bfs : $x_N = 0$; hence

$$B x_B + N \overset{0}{\overset{||}{x_N}} = b .$$

$$B x_B = b$$

$$x_B = B^{-1} b .$$

- Step 2: to find adjacent bfs formulate z and x_B in terms of x_N:

$$z(\text{current } bfs) = c_B^T x_B + \underbrace{c_N^T x_N}_{\overset{||}{0}} = c_B^T B^{-1} b$$

1) x_B:

$$B x_B + N x_N = b$$

$$B x_B = b - N x_N \qquad \times B^{-1}$$

$$x_B = B^{-1}(b - N x_N) = B^{-1} b - B^{-1} N x_N$$

2)

$$z = c_B^T x_B + c_N^T x_N = c_B^T B^{-1} b - \underbrace{c_B^T B^{-1} N x_N + c_N^T x_N}$$

$$z + (c_B^T B^{-1} N - c_N^T) x_N = c_B^T B^{-1} b .$$

$\hat{c}_N^T$

Reduced cost

$z + \hat{c}_N^T x_N = c_B^T B^{-1} b.$

Reduced cost of a non-basic variable:

(for the min problem!)

- If all $c_N^T \leq 0$, then z can/[✓ cannot be improved]

Hence $z = c_B^T B^{-1} b$ is optimal value

BFS - opt. solution

STOP

- There is $c_{N_t}^T > 0$: then z [✓ can]/cannot be improved

Reduced cost

- There is $c_{N_t}^T > 0$: decide what variable is entering/leaving:

1. Select non-basic variable t with the largest/smallest $\hat{c}_t$ among all $\widehat{c_N}$ > 0
2. Ratio Test:
   i. express x_B in terms of x_N:
   ii. Decide what component of x_B is leaving:

$$x_B = B^{-1}b - B^{-1}N x_N \rightarrow \text{all } x_N = 0 \text{ except for } (x_N)_t$$

$$x_B = \underbrace{B^{-1}b}_{\hat{b}} - \overbrace{B^{-1}A_t}^{\hat{A}_t}(x_N)_t$$

$\uparrow$

column corresponding to $(x_N)_t$

$x_B = \hat{b} - \hat{A}_t (x_N)_t \geq 0$ for all i in x_B

$(x_B)_i = \hat{b}_i - \hat{a}_{it} (x_N)_t \geq 0$

1. if all $\hat{a}_{it} \leq 0$ → can Increase $(x_N)_t$ indefinately → LP is unbounded

10:04

2. assume $\hat{a}_{it} > 0$

$$\hat{b}_i - \hat{a}_{it} (x_N)_t \geq 0 \rightarrow (x_N)_t \leq \frac{\hat{b}_i}{\hat{a}_{it}}$$

$$\downarrow$$

$$(x_N)_t = \min \left\{ \frac{\hat{b}_i}{\hat{a}_{it}} , \hat{a}_{it} > 0 \right\}$$

entering Basis

$$(x_B)_s : \quad s = \arg\min \left\{ \frac{\hat{b}_i}{\hat{a}_{it}} , \hat{a}_{it} > 0 \right\}$$

Simplex Method in algebraic form summary

- $\min\ (or\ max)\ z = c^T x$

  $s.t.\quad Ax = b,$

  $x \geq 0,$

  Standard form

where $b \geq 0$.

- Obtain the initial bfs; → $x_{B_0} \mid x_{N_0}$
- Compute the vector $\widehat{c_N^T} =$ $c_B^T B^{-1} N - c_N^T$ → vector of reduced costs for x_N
  - If $\widehat{c_N^T} \leq 0$ $(\widehat{c_N^T} \geq 0$), then the bfs is optimal STOP

    For min max
  - Otherwise select x_t with $\widehat{c_{N_t}^T} > 0$ which is the most positive → min problem

    (with $\widehat{c_{N_t}^T} < 0$ which is the most negative → max problem
- Compute $\widehat{A_t} =$ $B^{-1} A_t$, $\widehat{b_t} = B^{-1} b$. If $\widehat{A_t} \leq 0$, the LP is unbounded
- Otherwise chose $s =$ $\operatorname{argmin}\left\{ \frac{\hat{b}_t}{\hat{a}_{it}} ,\ \hat{a}_{it} > 0 \right\}$
- Select $(x_B)_s$ as the leaving component.

Simplex Method in tableau form

- Tableau form:

basis	$\mathbf{x}$	rhs
z	$\mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{A} - \mathbf{c}^T$ $= \hat{\mathbf{c}}^T$	$\mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{b}$
$\mathbf{x}_B$	$\mathbf{B}^{-1}\mathbf{A}$ $= \hat{\mathbf{A}}$	$\mathbf{B}^{-1}\mathbf{b}$ $= \hat{\mathbf{b}}$

R_0

$$\hat{c}^T = c_B^T B^{-1} A - c^T =$$
$$= c_B^T B^{-1}(B \mid N) - (c_B^T \mid c_N^T) =$$
$$= (c_B^T B^{-1} B - c_B^T \mid c_B^T B^{-1} N - c_N^T),$$
$$= (0^T \mid \hat{c}_N^T)$$

- Decomposition of the tableau on x_B and x_N:

canonical form

basis	$\mathbf{x}_N$	$\mathbf{x}_B$	rhs
z	$\underbrace{\mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_N^T}_{\hat{c}_N}$	$\underbrace{\mathbf{0}^T}_{\hat{c}_B}$	$\mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{b}$
$\mathbf{x}_B$	$\mathbf{B}^{-1}\mathbf{N}$	$\mathbf{I}$	$\mathbf{B}^{-1}\mathbf{b}$

$$B^{-1} A = B^{-1}(B \mid N) = (B^{-1} B \mid B^{-1} N) =$$
$$= (I \mid B^{-1} N)$$

$= (I \mid B \ N)$

Example

(3, 5)

not active
(non-binding)

$-6 + 5 < 2$ −

$-3 + 10 = 7$
$3 = 3$.

} active

- Solve:

$$\begin{aligned} \min z = & -x_1 - 2x_2 \\ s.t. \quad & -2x_1 + x_2 \le 2 \\ & -x_1 + 2x_2 \le 7 \\ & x_1 \le 3 \\ & x_1,\ x_2 \ge 0 \end{aligned}$$

non-binding

- Standard form:

$$\begin{aligned} \min z = & -x_1 - 2x_2 \\ s.t. \quad & -2x_1 + x_2 + s_1 = 2 \\ & -x_1 + 2x_2 + s_2 = 7 \\ & x_1 + s_3 = 3 \\ & x_1,\ x_2,\ s_1\ s_2\ s_3 \ge 0 \end{aligned}$$

Example

- Standard form: $\min z = -x_1 - 2x_2$

$$\begin{array}{lrcl} s.t. & -2x_1 + x_2 + s_1 & = & 2 \\ & -x_1 + 2x_2 + s_2 & = & 7 \\ & x_1 + s_3 & = & 3 \\ & x_1,\ x_2,\ s_1\ s_2\ s_3 & \geq & 0 \end{array}$$

$$c^T = (-1, -2, 0, 0, 0)$$

$$b = \begin{pmatrix} 2 \\ 7 \\ 3 \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix}$$

$$A = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$c^T = (\overbrace{-1, -2}^{c_N^T}, \overbrace{0, 0, 0}^{c_B^T})$$

$$X_{B_0}^T = (s_1 \quad s_2 \quad s_3)$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = B^{-1}$$

$$B^{-1}N = N = \begin{pmatrix} -2 & 1 \\ -1 & 2 \\ 1 & 0 \end{pmatrix} \qquad B^{-1}b = b = \begin{pmatrix} 2 \\ 7 \\ 3 \end{pmatrix}$$

$$C_B^T = (0, 0, 0) \rightarrow z = C_B^T B^{-1} N = 0.$$

$$\hat{C}_N^T = C_B^T B^{-1} N - C_N^T =$$

Initial tableau: $= -C_N^T = -(-1, -2)$

	x_1	x_2	s_1	s_2	s_3	RHS
z	1	2	0	0	0	0
s_1	-2	1	1	0	0	2
s_2	-1	2	0	1	0	7
s_3	1	0	0	0	1	3

$\underbrace{x_1 \quad x_2}_{x_N}$

x_2 enters

Ratio Test

2/1 → s_1 is leaving

7/2

z	5	0	-2	0	0	-4
x_2	-2	1	1	0	0	2
s_2	$\boxed{3}$	0	-2	1	0	3
s_3	1	0	0	0	1	3

x_1 enter

$R_0' = R_0 - 2R_1$

$R_1' = R_1$

Ratio

3/3 $R_2' = R_2 - 2R_1$

→ s_2 is leav

3/1 $R_3' = R_3$

	x_1	x_2	s_1	s_2	s_3	RHS	
z	0	0	$4/3$	$-5/3$	0	-9	$-5R_2'$
x_2	0	1	$-1/3 < 0$	$2/3$	0	4	$R_1' = R_1' + 2R_2'$
x_1	1	0	$-2/3 < 0$	$1/3$	0	1	$R_2' = \frac{R_2}{3}$
s_3	0	0	$\boxed{2/3}$	$-1/3$	1	2	$R_3' = R_3 - R_2'$

s_1 enters

s_3 leaving

z	0	0	0	-1	-2	-13	$-2R_3$
x_2	0	1	0	$1/2$	$1/2$	5	
x_1	1	0	0	0	1	3	
s_1	0	0	1	$-1/2$	$3/2$	3	

as all $\hat{c}_N = 0$,

$z^* = -13$ is optimal

$$x^* = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

$R_3' = \frac{3}{2} R_3$

$R_2' = R_2 + R_3$

$R_1' = R_1 + \frac{1}{2} R_3$

Example 2 – unique optimal solution

(*) HW

- Solve:

$$\min z = x_1 + x_2 - 4x_3$$

s.t.

$$\begin{aligned} x_1 + x_2 + 2x_3 + x_4 \qquad\qquad &= 9 \\ x_1 + x_2 - x_3 \qquad + x_5 \qquad &= 2 \\ -x_1 + x_2 + x_3 \qquad\qquad + x_6 &= 4 \\ x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6 \geq 0 & \end{aligned}$$

Example 2 – unique optimal solution

Example 3 – unbounded problem

- Standard form: $\max z = 2x_1 + 3x_2$

$$
\begin{aligned}
s.t. \quad & x_1 - x_2 + \boxed{s_1} = 1 \\
& x_1 - 2x_2 + \quad + \boxed{s_2} = 2 \\
& x_1,\ x_2, s_1, s_2 \geq 0
\end{aligned}
$$

$z - 2x_1 - 3x_2 = 0.$

$X_{B_0} = (s_1\ \ s_2)$

NOT optimal

	x_1	x_2	s_1	s_2	RHS
z	-2	-3	0	0	0
s_1	1	$\boxed{-1}$	1	0	1
s_2	1	$\boxed{-2}$	0	1	2

$A_t < 0 \rightarrow$

→

$x_1 - x_2 + s_1 = 1$

$x_1 - 2x_2 + s_2 = 2$

$x_1 = 0$ ↓

$s_1 = 1 + x_2$

$s_2 = 2 + 2x_2$

- This problem is unbounded.

Let x_1 enter

	x_1	x_2	s_1	s_2	RHS	Ratio
z	-2	-3	0	0	0	
s_1	1	-1	1	0	1	$1/1 \rightarrow$
s_2	1	-2	0	1	2	$2/1$
z	0	$\boxed{-5} < 0$	2	0	2	
x_1	1	-1	1	0	1	
s_2	0	-1	-1	1	1	

still unbounded

Example 4 – infinite number of optimal solutions

- Standard form:

$$\min z = -3x_1 - x_2 - \frac{1}{2}x_3$$

s.t.

$$6x_1 \quad -x_3 + s_1 = 12$$

$$x_2 + x_3 + \quad + s_2 = 2$$

$$x_1,\ x_2,\ x_3,\ s_1, s_2 \geq 0$$

- Solution in a tableau form:

basis	x_1	x_2	x_3	s_1	s_2	rhs
z	3	1	$\frac{1}{2}$	0	0	0
s_1	$\boxed{6}$	0	-1	1	0	12
s_2	0	1	1	0	1	10
z	0	1	1	$-\frac{1}{2}$	0	-6
x_1	1	0	$-\frac{1}{6}$	$\frac{1}{6}$	0	2
s_2	0	$\boxed{1}$	1	0	1	10
z	0	0	0 N	$-\frac{1}{2}$ N	$-\frac{1}{2}$ N	-16
x_1	1	0	$-\frac{1}{6}$	$\frac{1}{6}$	0	2
x_2	0	1	$\boxed{1}$	0	1	10
z	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-16
x_1	1	$\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{11}{3}$
x_3	0	1	1	0	1	10

opt

differa

Convergence and degeneracy of the Simplex Method

- Consider: $$\min z = -\frac{3}{4}x_1 + 150x_2 - \frac{1}{50}x_3 + 6x_4$$

s.t.

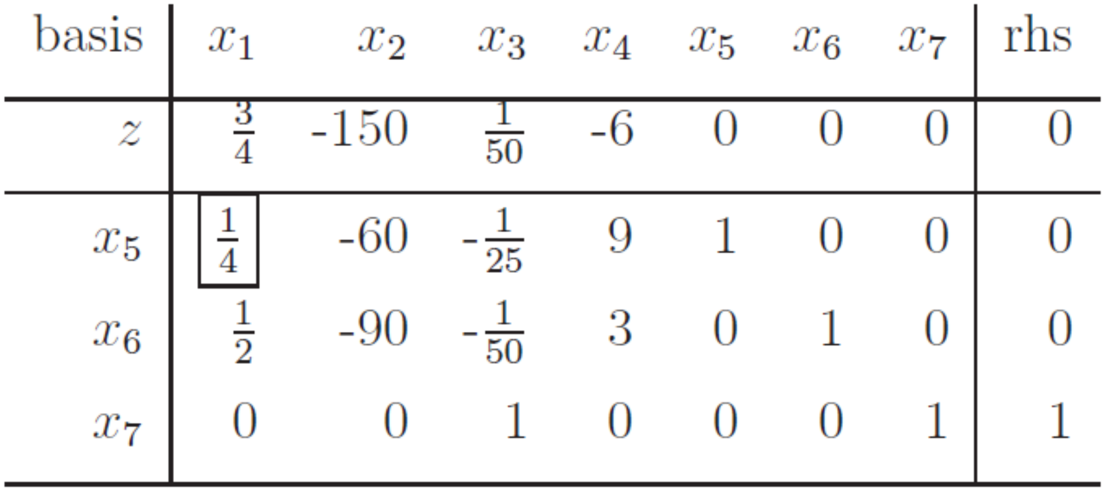
$$\frac{1}{4}x_1 - 60x_2 - \frac{1}{25}x_3 + 9x_4 + x_5 = 0$$

$$\frac{1}{2}x_1 - 90x_2 - \frac{1}{50}x_3 + 3x_4 + x_6 = 0$$

$$x_3 + x_7 = 1$$

$$x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6,\ x_7 \geq 0$$

- Initial tableau:

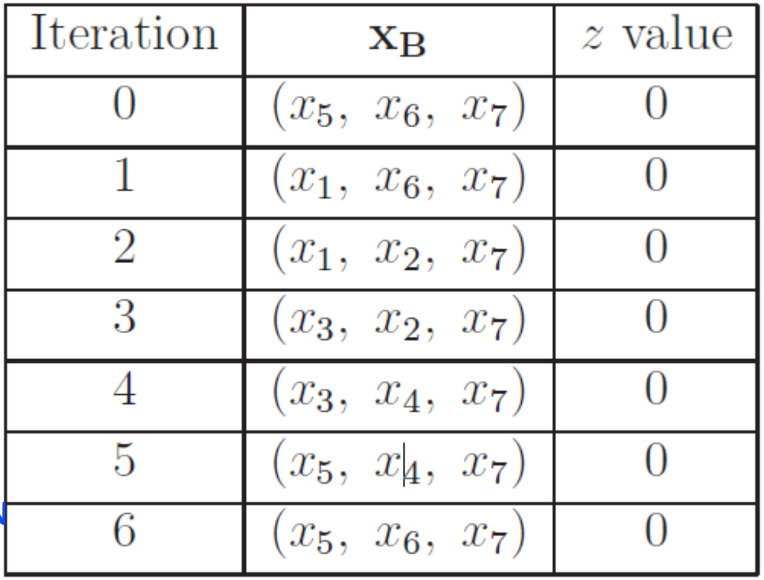


basis	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs
z	$\frac{3}{4}$	-150	$\frac{1}{50}$	-6	0	0	0	0
x_5	$\boxed{\frac{1}{4}}$	-60	$-\frac{1}{25}$	9	1	0	0	0
x_6	$\frac{1}{2}$	-90	$-\frac{1}{50}$	3	0	1	0	0
x_7	0	0	1	0	0	0	1	1

Convergence and degeneracy of the Simplex Method

- After six iterations:

Cycling!



Iteration	$\mathbf{x_B}$	z value
0	$(x_5,\ x_6,\ x_7)$	0
1	$(x_1,\ x_6,\ x_7)$	0
2	$(x_1,\ x_2,\ x_7)$	0
3	$(x_3,\ x_2,\ x_7)$	0
4	$(x_3,\ x_4,\ x_7)$	0
5	$(x_5,\ x_4,\ x_7)$	0
6	$(x_5,\ x_6,\ x_7)$	0

- To avoid cycling rhs may be changed:

$$b' = (0.0000001274,0.00000000432,1)^T$$

Example

- Solve:

$$
\begin{aligned}
&\min z = 2x_1 + 3x_2 \\
s.t.\quad & \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 \le 4 \\
& x_1 + 3x_2 \ge 20 \\
& x_1 + x_2 = 10 \\
& x_1,\ x_2 \ge 0
\end{aligned}
$$

- Standard form:

original problem (I)

$$
\begin{aligned}
&\min z = 2x_1 + 3x_2 \\
s.t.\quad & \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 + s_1 = 4 \\
& x_1 + 3x_2 - e_2 = 20 \\
& x_1 + x_2 \quad ? \quad = 10 \leftarrow \\
& x_1,\ x_2,\ s_1,\ e \ge 0
\end{aligned}
$$

NO unique var

- Issues:
  - *The initial solution*
  - *The constraints*

$(s_1\ e_2) \rightarrow (4, -20) \rightarrow$ NOT feasible

constr. (3) is not satisfied

Example – addressing the issues

IDEA!

- Introduce artificial variables: to address initial BFS issue

modified problem (II)

$$\min z = 2x_1 + 3x_2$$

$$\begin{array}{lrl}
\text{s.t.} & \frac{1}{2}x_1 + \frac{1}{4}x_2 + s_1 & = 4 \\
& x_1 + 3x_2 - e_2 + a_2 & = 20 \\
& x_1 + x_2 + a_3 & = 10 \\
& x_1,\ x_2,\ s_1\ e_2\ a_2\ a_3 & \geq 0
\end{array}$$

- ***Issues again:*** optimal solution for (II) is [redacted]. Is it feasible for (I)?

$x_B = (s_1\ a_2\ a_3)$ BFS $= (0, 0, 4, 0, 20, 10)$

constr. (3) is not satisfied

Example – addressing the issues

- Let’s introduce “Big $\boldsymbol{M}$”:
- $min\, z = 2x_1 + 3x_2 + Ma_2 + Ma_3$ (II)

$$
\text{s.t.} \quad
\begin{aligned}
\tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 + s_1 &= 4 \\
x_1 + 3x_2 - e_2 + a_2 &= 20 \\
x_1 + x_2 + a_3 &= 10 \\
x_1,\ x_2,\ s_1\ e_2\ a_2\ a_3 &\geq 0
\end{aligned}
$$

Big M method

1. Make all $rhs \geq 0$;
2. Add slack/substract excess variables to make equality constraints;
3. For each constraint i *without slack* add an artificial variable a_i;
4. For each a_i add Ma_i for min problems and subtract Ma_i for max problems.
5. Solve the modified problem (II) with Simplex method.

Big M method

- If in optimal solution for (II) all $a_i =$ 0, then it is optimal for (I)
- If in optimal solution for (II) at least one $a_i >$ 0, then it is Infeas. for (I) and (I) Infeasible
- If (II) is unbounded and all $a_i =$ 0 then (I) is Infeas/unbounded
- If (II) is unbounded and one or more $a_i >$ 0 then (I) is either infeasible or unbounded

11:45

||

Example 1

- $min\, z = 2x_1 + 3x_2$

$$s.t. \qquad \frac{1}{2}x_1 + \frac{1}{4}x_2 \le 4$$

$$x_1 + 3x_2 \ge 36$$

$$x_1 + x_2 = 10$$

$$x_1,\ x_2 \ge 0$$

Example 1