Introduction to Optimisation:

Linear Programming: Basics. Introduction to Simplex method

Lecture 2

Lecture notes by Dr. Julia Memar and Dr. Hanyu Gu and with an acknowledgement to Dr.FJ Hwang and Dr.Van Ha Do

Standard form

→ Linear program

An LP must be presented in the standard form, if we wish to use the *Simplex method*

1. all constraints are in the form of equations
2. all variables are nonnegative
3. all rhs are nonnegative

OF

$$\max\ z \text{ (or } \min z) = cx_1 + c_2x_2 + \cdots + c_nx_n$$

constraints

$$\begin{aligned} s.t. \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \\ & x_1,\ x_2, \dots, x_n \geq 0 \end{aligned}$$

$b_i \geq 0.$

Nonnegativity of Decision Variables

- Any urs variable x can be presented as $x = p - q$, where $p, q \geq 0$.

- Example:

$$\min z = 2x_1 + 30x_2$$

s.t.

$$4x_1 + 7x_2 \geq 1$$

$$8x_1 + 5x_2 \geq 3 \quad (*)$$

$$6x_1 + 9x_2 \geq -2$$

$$x_1,\ x_2\ urs$$

unrestricted in sign

Set $x_1 = p_1 - q_1$ and $x_2 = p_2 - q_2$ .The equivalent LP (**):

where $p_i \geq 0$ and $q_i \geq 0$

$$\min z' = 2p_1 - 2q_1 + 30p_2 - 30q_2$$

s.t.

$$4p_1 - 4q_1 + 7p_2 - 7q_2 \geq 1$$

$$8p_1 - 8q_1 + 5p_2 - 5q_2 \geq 3$$

$$6p_1 - 6q_1 + 9p_2 - 9q_2 \geq -2$$

$$p_1\ p_2\ q_1\ q_2 \geq 0$$

$$4p_1^* - 4q_1^* + 7p_2^* - 7q_2^* \geq 1$$

$$4(p_1^* - q_1^*) + 7(p_2^* - q_2^*) \geq 1$$

$$4\ x_1^* + 7\ x_2^* \geq 1$$

Nonnegativity of Decision Variables

- Show how to construct a solution for the original problem (*) using an optimal solution for the equivalent problem (**) – we assume that it exists.
- Will the constructed solution be optimal for (*)?

Let p_1^* p_2^* q_1^* q_2^* - be the optimal s-n for the equivalent problem (**)

Then $x_1^* = p_1^* - q_1^*$

$x_2^* = p_2^* - q_2^*$

① $x^* = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}$ is feasible due to feasibility of (p^*, q^*)

② Assume x^* is not optimal, and there exists $x' \to$ optimal : $z(x') < z(x^*)$

construct p', q' :

if $x_i' < 0 \to p_i' = 0$; $q_i' = -x_i'$;

$x_i' = p_i' - q_i'$; $p_i \geq 0$ $q_i \geq 0$.

$x_i' > 0 \to p_i' = x_i'$; $q_i' = 0$

$\downarrow$

$z'(p_1' \; p_2' \; q_1' \; q_2') < z'(p_1^* \; p_2^* \; q_1^* \; q_2^*)$

$\times$ as $(p_1^* \; p_2^* \; q_1^* \; q_2^*)$ is optimal

Slack and Surplus Variables

- Any inequality constraint can be converted into an equality constraint by adding slack or subtracting surplus nonnegative variables:

slack variable

$$x_1 - 2x_2 \leq 3 \qquad \Longrightarrow \qquad x_1 - 2x_2 + S_1 = 3$$

$$x_1,\ x_2 \geq 0 \qquad \Longrightarrow \qquad x_1,\ x_2,\ S_1 \geq 0$$

Excess variable

$$2x_1 + x_2 \geq 3 \qquad \Longrightarrow \qquad 2x_1 + x_2 - e_1 = 3$$

$$x_1,\ x_2 \geq 0 \qquad \Longrightarrow \qquad x_1,\ x_2,\ e_1 \geq 0$$

$2x_1 + x_2 = -2$

Standard form - summary

To bring an LP to the standard form:

- **Objective function:** if you wish to change objective function from minimisation (or maximisation) form, multiply it by -1 to convert the objective to a maximisation (or minimisation) one.

- **Constraints:** Convert any inequality to an equality constraint by the addition of slack or surplus variables (as appropriate).

  both sides

- **RHS:** If any rhs b_i is negative, multiply the whole constraint by -1.

- **Variables:** Any $urs\ x_j$ can be replaced by two nonnegative variables x'_j and $x"_j$:

$$x_j = x'_j - x"_j$$

What if I have

$x_i \leq 0 \rightarrow x_i' = -x_i$

$x_i' \geq 0$

Standard form

$\min z = 2x_1 + 30x_2$

$c = \begin{pmatrix} 2 \\ 30 \end{pmatrix}$

$c^T = (2 \ , \ 30)$

The LP problem in the standard form:

$$\max\ z \text{ (or } \min z) = c^T x$$

$$s.t.\ Ax = b,$$

$(1\times n) \times (n \times 1)$

$$x \geq 0$$

$c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}_{n\times 1}$ $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}_{n\times 1}$

$\underset{m\times n}{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & \vdots & & & \\ \vdots & \vdots & & & \\ a_{m1} & a_{m2} & & \cdots & a_{mn} \end{pmatrix}$

where x and c are n –dimensional vectors, A is an $m \times n$ matrix, and b is an m-dimensional vector. Note that $b \geq 0$.

$b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}_{m\times 1}$

vector of RHS

Standard form - assumptions

➢ We <u>assume</u> that $\underbrace{(A|b)}$ is *consistent* , that is that after application of Gaussian–Jordan method there are no rows $[0\ 0\ 0\ \ldots 0|c]$; $c \neq 0$

$(I \mid A^{-1} b)$

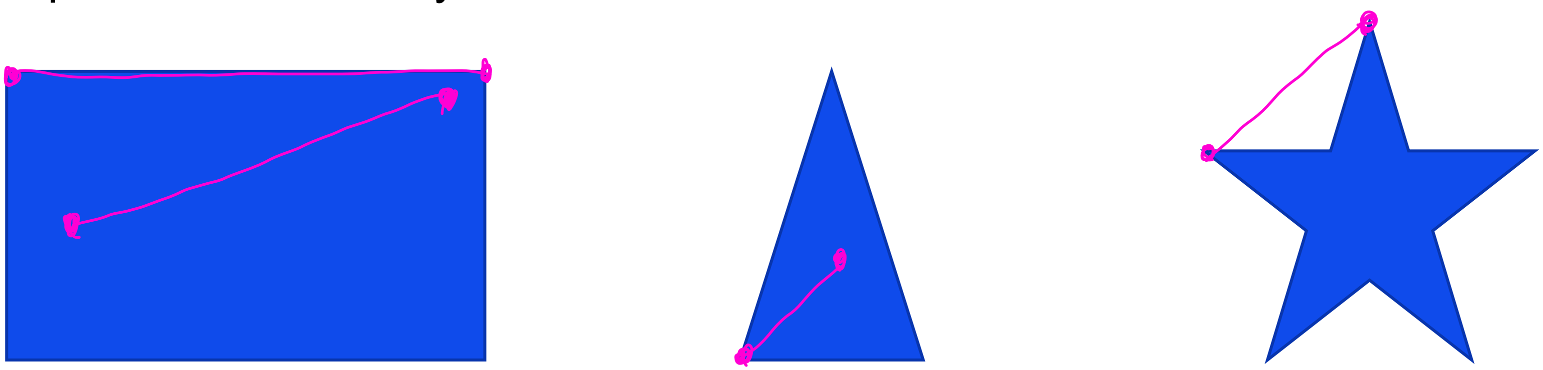
➢ If $n > m,$ then the number of variables is greater than the number of constraints,

Then the system has n - m degrees of freedom.

- Give an example of an LP with consistent $(A|b)$ and $n > m$ that's is infeasible (without a feasible solution) → HW

Fundamental Law of LP: definitions

- ***Convex set:*** A set S in n-dimensional space is *convex* if for any two points x_1 and x_2 from S any point of the line segment connecting x_1 and x_2 also belongs to S. In other words, a set S is a convex set if the line segment joining any pair of points in S is wholly contained in S.



convex ? yes

convex ? yes

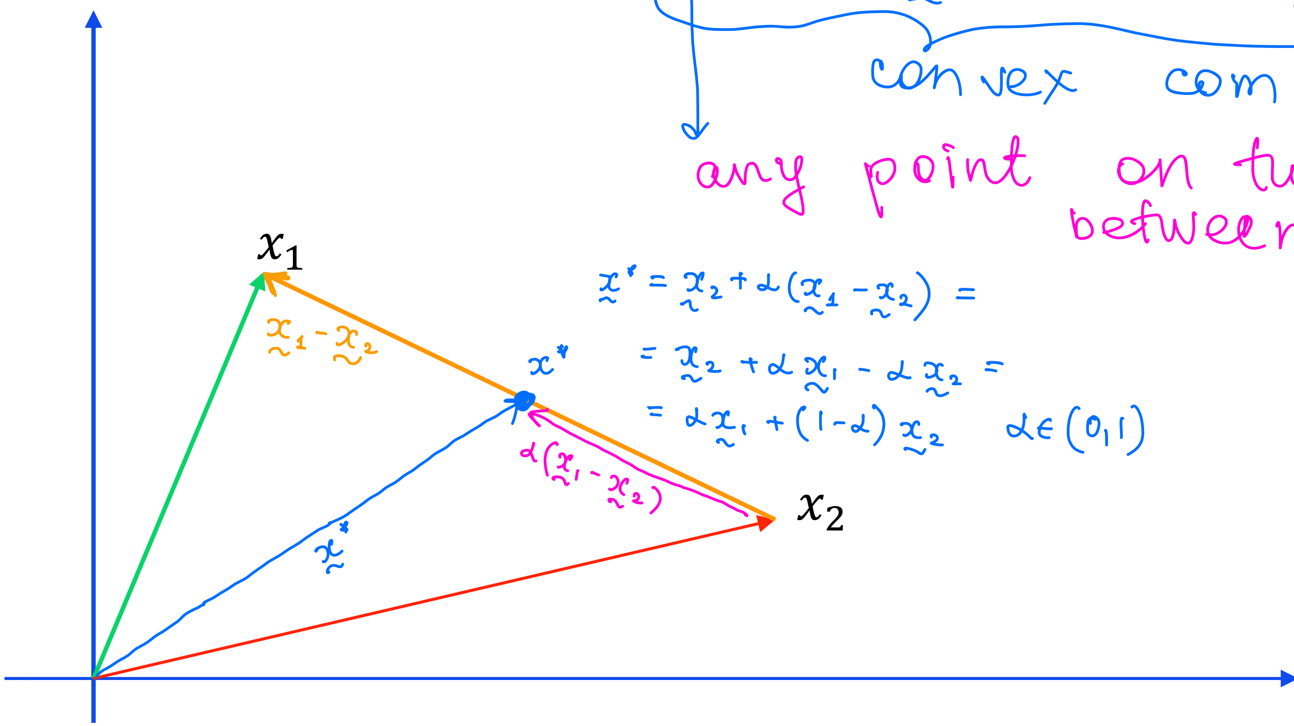
convex ? NO

Fundamental Law of LP: definitions

- ***Convex set:*** For any two points x_1 and x_2 in S and $\alpha \in (0,1)$, $x^* = \alpha x_1 + (1-\alpha)x_2 : x^* \in S$

$$\underbrace{x^* = \alpha \underset{\sim}{x}_1 + (1-\alpha)\underset{\sim}{x}_2}_{\text{convex combination}}$$

any point on the interval between x_1 and x_2



$$\underset{\sim}{x}^* = \underset{\sim}{x}_2 + \alpha(\underset{\sim}{x}_1 - \underset{\sim}{x}_2) =$$

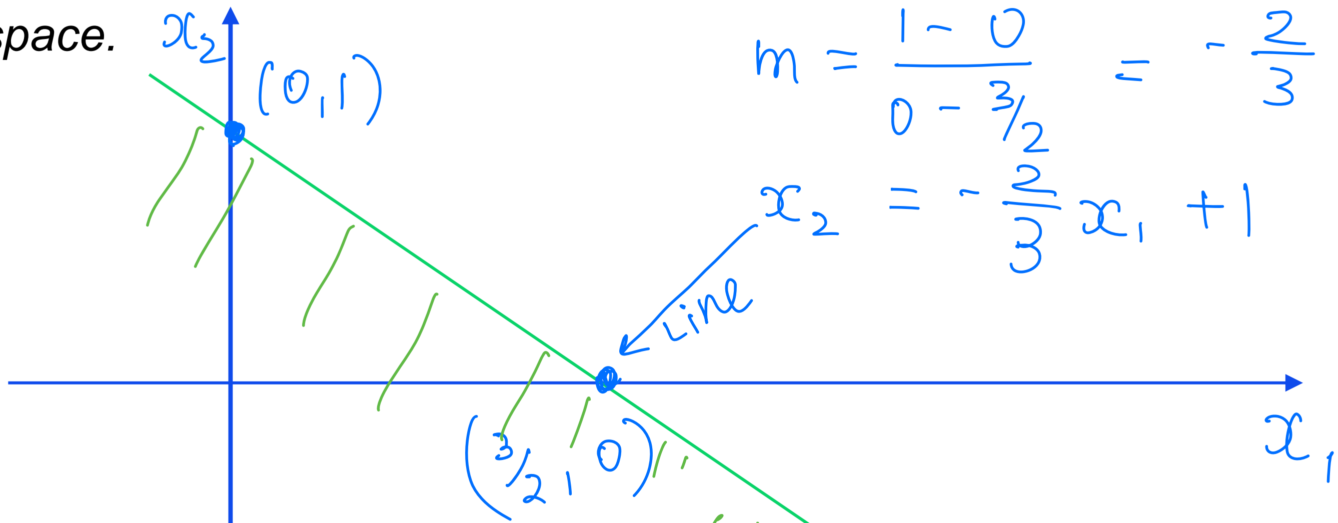
$$= \underset{\sim}{x}_2 + \alpha \underset{\sim}{x}_1 - \alpha \underset{\sim}{x}_2 =$$

$$= \alpha \underset{\sim}{x}_1 + (1-\alpha)\underset{\sim}{x}_2 \quad \alpha \in (0,1)$$

Fundamental Law of LP: definitions

- ***Closed half-space:*** for a given an n-dimensional row vector a and a constant b, a $closed\ half - space$ is the set of all vectors (or points) x in n −dimensional space satisfying $ax \leq b$.

- The set of vectors for which $ax = b$ is called ***the boundary*** *of the closed half-space.*



$$m = \frac{1-0}{0-3/2} = -\frac{2}{3}$$

$$x_2 = -\frac{2}{3}x_1 + 1$$

$$x_2 + \frac{2}{3}x_1 \leq 1 \rightarrow \text{Closed half-space}$$

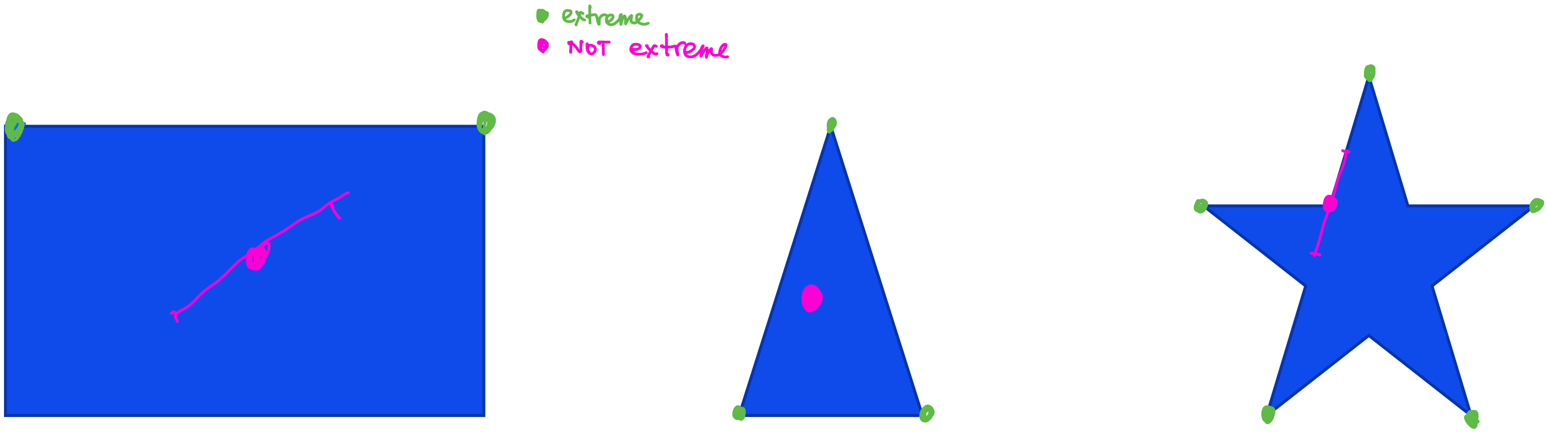
$$\begin{pmatrix} \frac{2}{3}, 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq 1$$

Fundamental Law of LP: definitions

- ***Extreme point:*** Given a convex set S of n −dimensional vectors, a point x^* is called an *extreme point* (or a corner point) of S if there are no two points x_1 and x_2 in S and a value $\alpha \in (0,1)$, such that

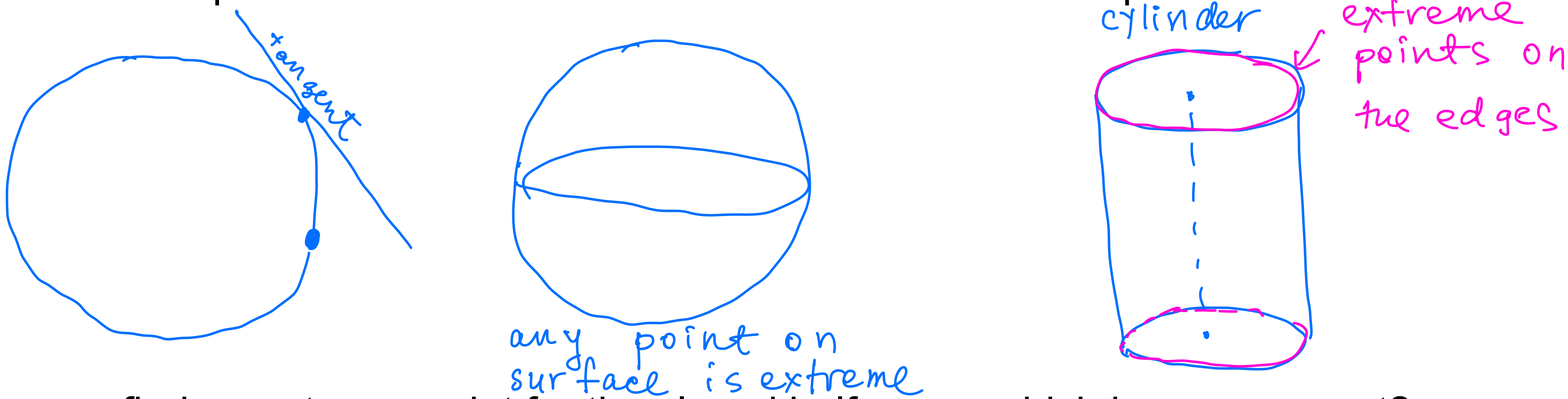
$$x^* = \alpha x_1 + (1 - \alpha) x_2$$

Or any line segment which lies in S and contains x^* has x^* as its end point.

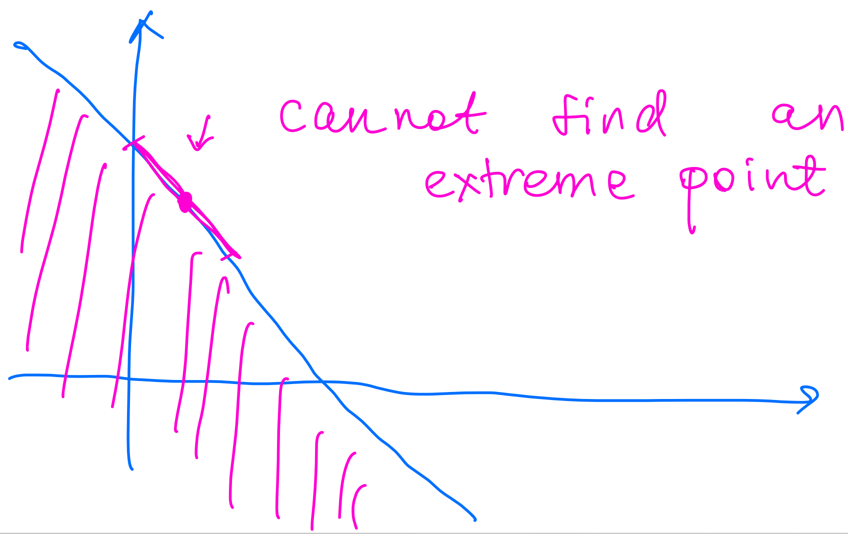


Exercise

- Give an example of convex set with infinite number of extreme points



- Can you find an extreme point for the closed half space which is a convex set?



Fundamental Law of LP: main results

- *Lemma 1 Every closed half-space is a convex set.*

Let S be a closed half-space:

any $x \in S$ $\quad ax \le b$

in

Let $x_1 \in S$; $x_2 \in S$:

$ax_1 \le b$ $\qquad ax_2 \le b$

Let $x^* = \alpha x_1 + (1-\alpha) x_2$; $\alpha \in (0,1)$

$ax^* = \alpha a x_1 + (1-\alpha) a x_2 \le \alpha b + (1-\alpha) b = b$

$ax^* \le b \rightarrow x^* \in S \rightarrow S$ is convex

- *Lemma 2* The intersection of any collection of convex sets is a convex set.

Prove lemma 2 (at home)

Fundamental Law of LP: main results

- ***Theorem 1*** *The feasible set of an LP problem is convex (assuming empty set is convex).*

1. by Lemma 1 every LP constraint is a closed-half space is convex

2. Feas. region is the overlap of all closed-half spaces that represent the constraints by lemma 2, Feas. region is convex (as intersection of convex sets)

Fundamental Law of LP: main results

For all results below we assume that an LP in a standard form

- *Lemma 3* If $x = 0$ is a feasible solution of an LP, then it is an extreme point.
- *Lemma 4* For an LP, $x \neq 0$ is an extreme point if and only if the columns of A corresponding to the non-zero x_i are linearly independent.
- *Lemma 5* If the LP is feasible, then it has an extreme point.
- ***Theorem 2*** *The feasible region for any LP has a finite number of extreme points.*

$$\underbrace{n^{50} \quad 2^{50}}_{\text{Finite Number}}$$

For any extreme point There a set of linear-Ind columns of A.

$\downarrow$

$$N_{\text{Extreme points}} \leq N_{\text{of combinations}} = {}_nC^1 + {}_nC^2 + {}_nC^3 + \dots {}_nC^n$$

$${}_nC^k = \frac{n!}{k!(n-k)!}$$

$\uparrow$ chose k out of n

$1 \le k \le n$

Fundamental Law of LP: main results

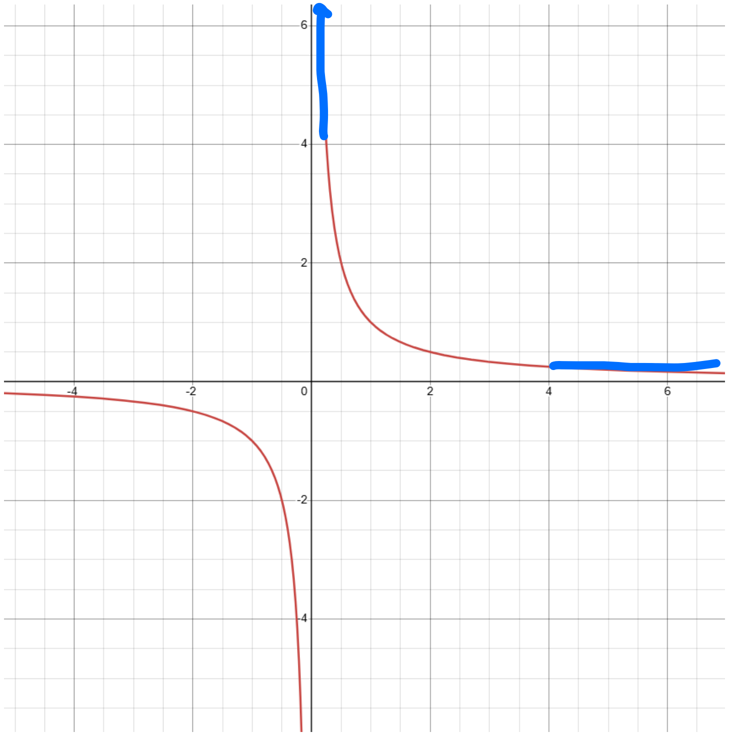
- ***Theorem 3*** If the feasible set is non-empty and one optimal solution exists to the LP, then there is an optimal solution at one of the extreme points.

- For an LP, if objective function value is bounded, then optimal solution exists.

Fundamental Law of LP: main results

In general, there may be three cases for the type of optimal objective function value:

- Finite with at least one optimal solution
- Bounded but not obtainable (consider $\min \frac{1}{x} \quad x > 0$) $= 0$
- Unbounded (therefore no optimal solution, and may/may not have a convergent sequence, consider $\max \frac{1}{x} \quad x > 0$) ∞



Basic Feasible Solution (BFS)

$5x_1 + 2x_2 + x_3 = 1$

$3x_1 + x_2 - x_4 = 2$

$$A = \begin{array}{cccc} \begin{pmatrix} 5 & 2 & 1 & 0 \\ 3 & 1 & 0 & -1 \end{pmatrix} \\ \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ A_1 & A_2 & A_3 & A_4 \end{array} \end{array}$$

Consider an LP with constraints

$$Ax = b, \qquad (1)$$

$$x \geq 0 \qquad (2)$$

Assume that $n > m$, $rank(A) = m$, and the feasible region is not empty.

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_m \\ x_{m+1} \\ \vdots \\ x_n \end{pmatrix} \rightarrow x_{m+1} = \ldots = x_n = 0$$

Basic feasible solution:

- Set $n-m$ components of x, to zero.
- Hence if remaining m columns of A are linearly independent, then there exists unique solution.
- *Basic feasible solution is the* unique solution for the m components together with n- m zero components.
- The m components are called *basic variables* x_B and the zero components are called *non-basic variables* x_N.

$|x_B| = m$

$|x_N| = n-m$

Basic Feasible Solution (BFS)

B is comprised of the column corresponding to x_B ; N → to x_N

$A_{m \times n}$

- BFS:

$(x_1, x_2, x_3, \ldots, x_n)^T = (x_B \mid x_N)^T$, and $A = (\underbrace{B}_{m \times m} \mid \underbrace{N}_{m \times (n-m)})$

Then (1) can be presented as :

$$Ax = b$$

$$(B \mid N)\begin{pmatrix} x_B \\ x_N \end{pmatrix} = b$$

$$Bx_B + \underbrace{Nx_N}_{= 0} = b$$

as $x_N = 0$ for the current BFS

$$Bx_B = b \rightarrow x_B = \underbrace{B^{-1}}_{\text{Inverse of } B} b$$

For the current BFS:

$$z = c^T x = \left(c_B^T \mid c_N^T \right) \begin{pmatrix} x_B \\ x_N \end{pmatrix} =$$

$$= c_B^T \underbrace{x_B}_{B^{-1} b} + c_N^T \underbrace{x_N}_{= \emptyset} = c_B^T B^{-1} b$$

$$\boxed{z = c_B^T B^{-1} b}$$

Basic Feasible Solution (BFS)

Lemma 6 For an LP, x is an extreme point if and only if it is a basic feasible solution. It is assumed that the LP is in a standard form.

Theorem 4 *If the feasible set is non-empty and one optimal solution exists to the LP, then there is a basic feasible solution giving the optimal value.*

- *Degeneracy* – if more than one bfs represents the same extreme point of the feasible set – to be discussed later…..

Basic Feasible Solution (BFS)

- Example: $max\ 5x_1 + 4x_2$

  s.t. $3x_1 + 2x_2 \leq 120$

  $x_1 + x_2 \leq 50$

  $x_1, x_2 \geq 0$

- Standard form: $max\ 5x_1 + 4x_2 + 0s_1 + 0s_2$

  s.t. $3x_1 + 2x_2 + s_1 = 120$

  $x_1 + x_2 + s_2 = 50$

  $x_1, x_2, s_1, s_2 \geq 0$

$$A = \begin{array}{cccc} x_1 & x_2 & s_1 & s_2 \\ \end{array}$$
$$A = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

$n = 4$; $m = 2$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{pmatrix} \quad b = \begin{pmatrix} 120 \\ 50 \end{pmatrix}$$

$$c = \begin{pmatrix} 5 \\ 4 \\ 0 \\ 0 \end{pmatrix}$$

$|x_B| = 2$

$|x_N| = 2$

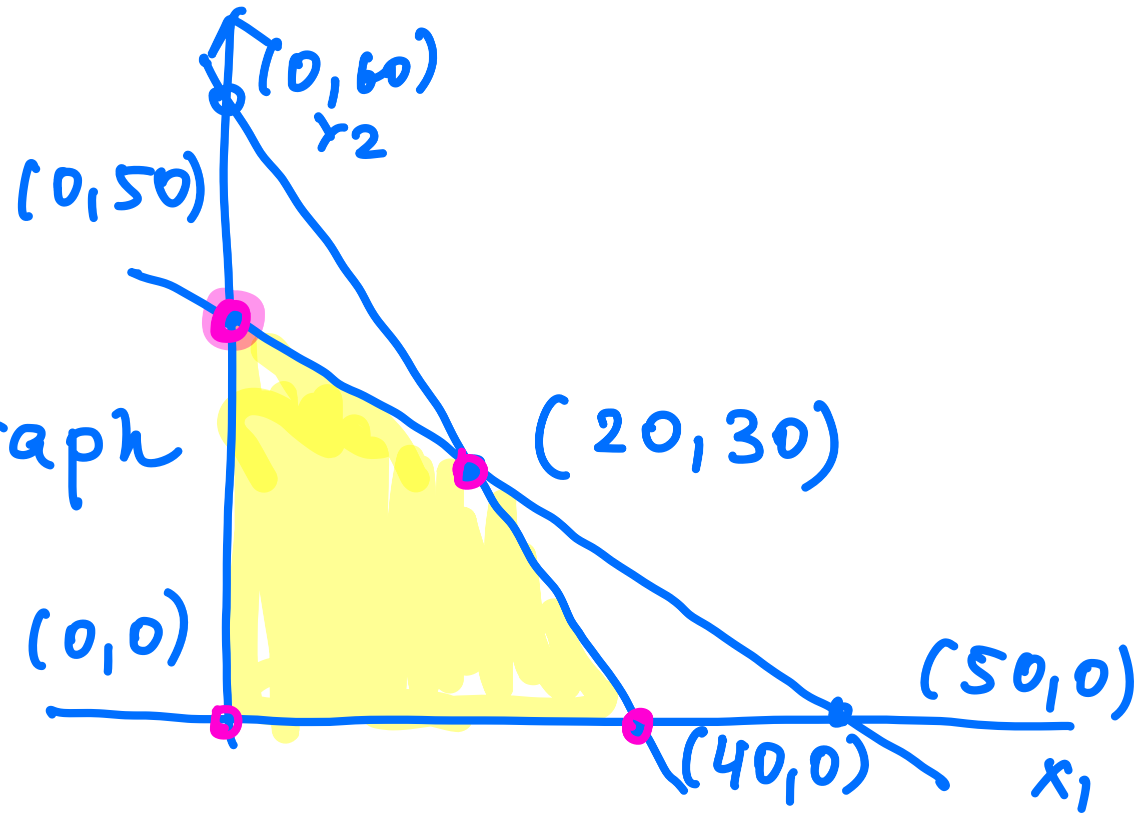
Basic Feasible Solution (BFS)

Possible combinations of potential bfs:

$${}^{2}_{4}C = \frac{4!}{2!\,2!} = 6$$

4 extreme points

6 points on the graph



$x_1 \quad x_2 \quad s_1 \quad s_2$

① $x_B = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = x_N \rightarrow$ point $(0,0)$

$$x_B = B^{-1} b \; ; \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = B^{-1} \rightarrow x_B = b = \begin{pmatrix} 120 \\ 50 \end{pmatrix}$$

BFS : $(0, 0, 120, 50) \longrightarrow$ extreme point

$$z = c_B^T B^{-1} b = (0, 0)\begin{pmatrix} 120 \\ 50 \end{pmatrix} = \emptyset$$

② $x_B = \begin{pmatrix} x_1 \\ s_2 \end{pmatrix} \rightarrow x_N = \begin{pmatrix} s_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = x_N$

$$B = \begin{array}{c} \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix} \\ \begin{array}{cc} \uparrow & \uparrow \\ x_1 & s_2 \end{array} \end{array} \rightarrow B^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$x_B = \underbrace{\frac{1}{3}\begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix}}_{B^{-1}} \underbrace{\begin{pmatrix} 120 \\ 50 \end{pmatrix}}_{b} = \begin{pmatrix} 40 \\ 10 \end{pmatrix}$$

BFS : $(40, 0, 0, 10) \rightarrow$ Feasible $\rightarrow$ point $(40,0)$

$$z(BFS) = c_B^T B^{-1} b = (5, 0)\begin{pmatrix} 40 \\ 10 \end{pmatrix} = 200$$

③ $x_B = \begin{pmatrix} x_1 \\ s_1 \end{pmatrix} \rightarrow x_N = \begin{pmatrix} s_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$B = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix} \quad B^{-1} = -1 * \begin{pmatrix} 0 & -1 \\ -1 & 3 \end{pmatrix}$

$x_B = \begin{pmatrix} 0 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 120 \\ 50 \end{pmatrix} = \begin{pmatrix} 50 \\ -30 \end{pmatrix} < 0$

B.S $= (50, 0, -30, 0) \rightarrow$ not a feasible s-n
point $(50, 0)$
not extreme point
of feas. region

④ $x_B = \begin{pmatrix} x_2 \\ s_1 \end{pmatrix} \rightarrow \begin{pmatrix} s_2 \\ x_1 \end{pmatrix} = x_N$

$B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$; $B^{-1} = -1 \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}$

$x_B = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}}_{B^{-1}} \underbrace{\begin{pmatrix} 120 \\ 50 \end{pmatrix}}_{b} = \begin{pmatrix} 50 \\ 20 \end{pmatrix}$

BFS = (0, 50, 20, 0) → feasible

point (0, 50) ← extreme point

$$Z(BFS) = \underset{c_B^T}{(4, 0)} \underset{B^{-1}b}{\begin{pmatrix} 50 \\ 20 \end{pmatrix}} = 200$$

5

$$x_B = \begin{pmatrix} x_2 \\ s_2 \end{pmatrix} \quad x_N = \begin{pmatrix} s_1 \\ x_1 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} ; \; B^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$$

$$x_B = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 120 \\ 50 \end{pmatrix} = \begin{pmatrix} 60 \\ -10 \end{pmatrix} < 0$$

BS = (0, 60, 0, -10) - not feasible

point (0, 60) - not extreme!

6

$$x_B = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \qquad x_N = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} \qquad B^{-1} = -1 \begin{pmatrix} 1 & -3 \\ -1 & 2 \end{pmatrix}$$

$$x_B = \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 120 \\ 50 \end{pmatrix} = \begin{pmatrix} 30 \\ 20 \end{pmatrix}$$

$$BFS = (20, 30, 0, 0) \text{ - feas.}$$

$$z(BFS) = (\underset{c_2}{4}, \underset{c_1}{5}) \begin{pmatrix} 30 \\ 20 \end{pmatrix} = 220 \text{ - max}$$

Basic Feasible Solution (BFS)

Feasible region: