Lecture Notes – Part 4

Big-M Method

Initial Basic Feasible Solution

Recall that performing the Simplex algorithm requires an initial basic feasible solution (bfs). All the LP problems we have solved so far are those with $\leq$ constraints and $\mathbf{b} \geq \mathbf{0}$, and hence we have found an initial bfs easily by using the slack variables as the initial basic variables. In other words, an initial bfs is obvious in the considered LP in standard form. Provided that we are faced with LPs involving any $\geq$ or equality constraint in general form, however, an initial bfs may not be readily apparent in standard form. The examples demonstrated in the following sections will illustrate that an initial bfs may be difficult to find. When a starting bfs is by no means obvious, the Big-M method or the two-phase Simplex method could be exploited to solve the problem.

The Big-M Method[1]

In this section, we discuss the Big-M method, a version of the Simplex algorithm that first finds an initial bfs by introducing dummy variables into the LP involving the $\geq$ or equality constraint(s) in general form. The ob-jective function of the original LP must, of course, be modified to ensure that the dummy variables are all equal to 0 at the conclusion of the Simplex algorithm. The following example illustrates the Big-M method.

[1]Section 4.12 in the reference book "Operations Research: Applications and Algorithms" (Winston, 2004)

Example 1

$$\begin{array}{rrcrcr} \min z = & 2x_1 & + & 3x_2 & & \\ s.t. & \frac{1}{2}x_1 & + & \frac{1}{4}x_2 & \leq & 4 \\ & x_1 & + & 3x_2 & \geq & 20 \\ & x_1 & + & x_2 & = & 10 \\ & x_1, x_2 \geq 0. & & & & \end{array}$$

To put the general form into standard form, we add a slack variable s_1 to the first constraint and subtract an excess/surplus variable e_2 from the second constraint.

$$\begin{array}{rrcrcrcrcr} \min z = & 2x_1 & + & 3x_2 & & & & & & \\ s.t. & \frac{1}{2}x_1 & + & \frac{1}{4}x_2 & + & s_1 & & & = & 4 \\ & x_1 & + & 3x_2 & & & - & e_2 & = & 20 \\ & x_1 & + & x_2 & & & & & = & 10 \\ & x_1, x_2, s_1, e_2 \geq 0. & & & & & & & & \end{array} \tag{1}$$

In searching for an initial bfs, we see that $s_1 = 4$ could be used as a basic (and feasible) variable for the row-1 equation. If we multiply row 2 by -1, we see that $e_2 = -20$ could be used as a basic variable for row 2. Unfortunately, $e_2 = -20$ violates the sign restriction $e_2 \geq 0$. Finally, in row 3 there is no readily apparent basic variable.[2] Thus, in order to use the Simplex method, each of rows 2 and 3 needs a basic variable which is feasible. To remedy this problem, we simply "invent" a basic feasible variable for each equation that needs one. Since these variables are created and are not real variables existing in the standard form of the original LP, we call them *artificial variables.* If an artificial variable is added to row i, we label it a_i. In the considered LP, we need to add an artificial variable a_2 to row 2 and an artificial variable a_3 to row 3. The resulting canonical form is

$$\begin{array}{rrcrcrcrcrcrcr} \min z = & 2x_1 & + & 3x_2 & & & & & & & & & & \\ s.t. & \frac{1}{2}x_1 & + & \frac{1}{4}x_2 & + & s_1 & & & & & & & = & 4 \\ & x_1 & + & 3x_2 & & & - & e_2 & + & a_2 & & & = & 20 \\ & x_1 & + & x_2 & & & & & & & + & a_3 & = & 10 \\ & x_1, x_2, s_1, e_2, a_2, a_3 \geq 0. & & & & & & & & & & & & \end{array} \tag{2}$$

We now have a bfs: $s_1 = 4, a_2 = 20, a_3 = 10$. Unfortunately, there is no guarantee that the optimal solution to (2) will be the same as the optimal

[2]Notice that row 3 does not satisfy the requirement of the canonical form.

solution to (1). In solving (2), we might obtain an optimal solution in which one or more artificial variables are positive. Such a solution may not be feasible in (1). For example, in solving (2), the optimal solution may easily be shown to be $s_1 = 4, a_2 = 20, a_3 = 10, x_1 = x_2 = e_2 = 0$. This "solution" obviously cannot possibly solve our original LP problem! To guarantee that the optimal solution to (2) is to solve (1), we must make sure that the optimal solution to (2) sets all artificial variables equal to zero. In a minimisation problem, we can ensure that all the artificial variables will be zero by adding a term Ma_i to the objective function for each artificial variable a_i. (In a maximisation problem, add a term $-Ma_i$ to the objective function.) Here M represents a "very large" positive number. Thus, we can amend (2) and have

$$
\begin{array}{rrcrcrcrcrcrcrcl}
\min z = & 2x_1 & + & 3x_2 & & & & & + & Ma_2 & + & Ma_3 & & \\
s.t. & \frac{1}{2}x_1 & + & \frac{1}{4}x_2 & + & s_1 & & & & & & & = & 4 \\
 & x_1 & + & 3x_2 & & & - & e_2 & + & a_2 & & & = & 20 \\
 & x_1 & + & x_2 & & & & & & & + & a_3 & = & 10 \\
 & \multicolumn{13}{l}{x_1, x_2, s_1, e_2, a_2, a_3 \geq 0.}
\end{array} \qquad (3)
$$

Rectifying the objective function in this way makes it extremely costly for an artificial variable to be positive. With this modified objective function, it seems reasonable that the optimal solution to (3) will have $a_2 = a_3 = 0$. In this case, the optimal solution to (3) will solve the original standard form (1). It sometimes happens, however, that in solving the modified formulation like (3), some of the artificial variables may assume positive values in the optimal solution. If this occurs, the original problem has no feasible solution. For obvious reasons, the method we have just outlined is often called the Big-M method.

Now we give the formal description of the Big-M method.

Step 1. Modify the constraints so that the rhs of each constraint is nonnegative. This requires that each constraint with a negative rhs be multiplied through by -1.[3]

Step 2. Convert the system of constraints to standard form, i.e. add a slack

[3] Remember that if you multiply an inequality by any negative number, the direction of the inequality is reversed. For example, we would transform the inequality $x_1 + x_2 \geq -1$ into $-x_1 - x_2 \leq 1$. The inequality $x_1 - x_2 \leq -2$ would be transformed into $-x_1 + x_2 \geq 2$.

variable s_i in the lhs of each $\leq$ constraint and subtract an excess variable e_i from the lhs of each $\geq$ constraint.

Step 3. For each constraint without a slack variable s_i, add an artificial variable a_i in the lhs. Also add the sign restriction $a_i \geq 0$.

Step 4. Let M denote a very large positive number. If the LP is a min problem, add (for each artificial variable) Ma_i to the objective function. If the LP is a max problem, add (for each artificial variable) $-Ma_i$ to the objective function.

Step 5. Because each artificial variable will be in the initial basis, all artificial variables must be eliminated from row 0 to make a canonical form before beginning the Simplex procedure. The choice of the entering variable depends on the multiplier of M since M is a very large positive number. For example, $4M-2$ is more positive than $3M+900$, and $-6M+5$ is more negative than $-5M-40$. Now solve the transformed problem by the Simplex method. If all artificial variables are equal to zero in the optimal solution, then we have found the optimal solution to the original LP. If any artificial variables are positive in the optimal solution to the transformed problem, then the original problem is infeasible.[4]

When an artificial variable leaves the basis, its column may be dropped from future tableaux. The purpose of an artificial variable is only to get an initial bfs. Once an artificial variable leaves the basis, we no longer need it.[5]

Now we resume solving the modified LP (3), which actually can be obtained by going through Steps 1–4. Going to Step 5, we first rearrange row 0,

$$z - 2x_1 - 3x_2 - Ma_2 - Ma_3 = 0,$$

to satisfy the canonical form. Because a_2 and a_3 are in the initial bfs, they must be eliminated from row 0. To eliminate a_2 and a_3 from row 0, simply

[4] We have ignored the possibility that when the modified LP (with the artificial variables) is solved, the final tableau may indicate that the LP is unbounded. If the final tableau indicates the LP is unbounded and all artificial variables in this tableau equal zero, then the original LP is unbounded. If the final tableau indicates that the LP is unbounded and at least one artificial variable is positive, then the original LP is infeasible.

[5] Despite this fact, we will maintain the artificial variables in all tableaux when doing Primal-Dual transformation which will be introduced in the following chapters.

replace row 0 by row 0 $+M$(row 2)$+M$(row 3). This yields the new row 0:

$$z + (2M-2)x_1 + (4M-3)x_2 - Me_2 = 30M.$$

Combining the new row 0 with rows 1–3 yields the initial tableau shown below.

basis	x_1	x_2	s_1	e_2	a_2	a_3	rhs	ratio test
z	$2M-2$	$4M-3$	0	$-M$	0	0	$30M$	
s_1	$\frac{1}{2}$	$\frac{1}{4}$	1	0	0	0	4	$\frac{4}{(\frac{1}{4})} = 16$
a_2	1	$\boxed{3}$	0	-1	1	0	20	$\frac{20}{3}*$
a_3	1	1	0	0	0	1	10	$\frac{10}{1} = 10$

We are solving a min problem, so the variable with the most positive coefficient in row 0 should enter the basis. Because $4M > 2M$, variable x_2 should enter the basis. The ratio test indicates that x_2 should enter the basis in row 2, which means the artificial variable a_2 will leave the current basis. The most difficult part of doing the pivot is eliminating x_2 from row 0. First, replace row 2 by $\frac{1}{3}$(row 2). Thus, the new row 2 is

$$\frac{1}{3}x_1 + x_2 - \frac{1}{3}e_2 + \frac{1}{3}a_2 = \frac{20}{3}.$$

We can now eliminate x_2 from row 0 by adding $-(4M-3)$(new row 2) to row 0. Then we have the new row 0:

$$z + \frac{2M-3}{3}x_1 + \frac{M-3}{3}e_2 + \frac{3-4M}{3}a_2 = \frac{10M+60}{3}.$$

After using EROs to eliminate x_2 from row 1 and row 3 as well, we obtain the following tableau.

basis	x_1	x_2	s_1	e_2	a_2	a_3	rhs	ratio test
z	$\frac{2M-3}{3}$	0	0	$\frac{M-3}{3}$	$\frac{3-4M}{3}$	0	$\frac{10M+60}{3}$	
s_1	$\frac{5}{12}$	0	1	$\frac{1}{12}$	$-\frac{1}{12}$	0	$\frac{7}{3}$	$\frac{(\frac{7}{3})}{(\frac{5}{12})} = \frac{28}{5}$
x_2	$\frac{1}{3}$	1	0	$-\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{20}{3}$	$\frac{(\frac{20}{3})}{(\frac{1}{3})} = 20$
a_3	$\boxed{\frac{2}{3}}$	0	0	$\frac{1}{3}$	$-\frac{1}{3}$	1	$\frac{10}{3}$	$\frac{(\frac{10}{3})}{(\frac{2}{3})} = 5*$

Because $\frac{2M}{3} > \frac{M}{3}$, we next enter x_1 into the basis. The ratio test indicates that a_3 in the third row shall leave the current basis. Then our next tableau will have $a_2 = a_3 = 0$. After the similar EROs, we have the following new tableau.

basis	x_1	x_2	s_1	e_2	a_2	a_3	rhs
z	0	0	0	$-\frac{1}{2}$	$\frac{1-2M}{2}$	$\frac{3-2M}{2}$	25
s_1	0	0	1	$-\frac{1}{8}$	$\frac{1}{8}$	$-\frac{5}{8}$	$\frac{1}{4}$
x_2	0	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	5
x_1	1	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$	5

Because all variables in row 0 have nonpositive coefficients, this is an optimal tableau; all artificial variables are equal to zero in this tableau, so we have found the optimal solution to the original LP problem: $x_1 = 5, x_2 = 5, s_1 = \frac{1}{4}, e_2 = 0$ with $z_{\min} = 25$. Note that to obtain this optimal solution the a_2 column could have been dropped after a_2 left the basis (at the conclusion of the first pivot), and the a_3 column could have been dropped after a_3 left the basis (at the conclusion of the second pivot).

Now let's consider another example, which is obtained by modifying Example 1.

Example 2

$$\begin{array}{rrcrcr} \min z = & 2x_1 & + & 3x_2 & & \\ s.t. & \frac{1}{2}x_1 & + & \frac{1}{4}x_2 & \leq & 4 \\ & x_1 & + & 3x_2 & \geq & 36 \\ & x_1 & + & x_2 & = & 10 \\ & x_1, x_2 \geq 0. & & & & \end{array}$$

After going through Steps 1–5 of the Big-M method, we obtain the initial tableau below.

basis	x_1	x_2	s_1	e_2	a_2	a_3	rhs	ratio test
z	$2M-2$	$4M-3$	0	$-M$	0	0	$46M$	
s_1	$\frac{1}{2}$	$\frac{1}{4}$	1	0	0	0	4	$\frac{4}{(\frac{1}{4})} = 16$
a_2	1	3	0	-1	1	0	36	$\frac{36}{3} = 12$
a_3	1	$\boxed{1}$	0	0	0	1	10	$\frac{10}{1} = 10^*$

Because $4M \geq 2M$, we enter x_2 into the basis. The ratio test indicates that x_2 should be entered in row 3, causing a_3 to leave the basis. After entering x_2 into the basis, we obtain the tableau as follows.

basis	x_1	x_2	s_1	e_2	a_2	a_3	rhs
z	$1-2M$	0	0	$-M$	0	$3-4M$	$6M+30$
s_1	$\frac{1}{4}$	0	1	0	0	$-\frac{1}{4}$	$\frac{3}{2}$
a_2	-2	0	0	-1	1	-3	6
x_2	1	1	0	0	0	1	10

Because each variable has a nonpositive coefficient in row 0, this is an optimal tableau. The optimal solution indicated by this tableau is $s_1 = \frac{3}{2}, a_2 = 6, x_2 = 10, a_3 = e_2 = x_1 = 0$ with $z_{\min} = 6M + 30$. The artificial variable a_2 is positive in the optimal tableau, so Step 5 shows that the original LP has no feasible solution. In summary, if any artificial variable is positive in the optimal Big-M tableau, then the original LP has no feasible solution.

Note that when the Big-M method is used, it is difficult to determine how large M should be. Generally, M is chosen to be at least 100 times larger than the largest coefficient in the original objective function. The introduction of such large numbers into the problem can cause round-off errors and other computational difficulties. For this reason, most computer codes solve LPs by using the two-phase Simplex method.

3 The Two-Phase Simplex Method[6]

When a bfs is not readily available, the two-phase Simplex method can be used as an alternative to the Big-M method. The idea of this method is to solve an auxiliary LP to get an initial bfs for the original LP in Phase I, and then switch back to solve the original LP in Phase II.

In the two-phase Simplex method, we add artificial variables to the same constraints as we did in the Big-M method. Then we find a bfs to the original

[6] Section 4.13 in the reference book "Operations Research: Applications and Algorithms" (Winston, 2004)

LP by solving the Phase-I auxiliary LP. In the Phase-I LP, the objective function is to minimise the sum of all artificial variables. At the completion of Phase I, we reintroduce the original LP's objective function and determine the optimal solution to the original LP in Phase II. The following steps describe the two-phase Simplex method. Note that steps 1–3 for the two-phase Simplex are identical to those for the Big-M method.

Step 1. Modify the constraints so that the rhs of each constraint is nonnegative. This requires that each constraint with a negative rhs be multiplied through by -1.

Step 2. Convert the system of constraints to standard form, i.e. add a slack variable s_i in the lhs of each $\leq$ constraint and subtract an excess variable e_i from the lhs of each $\geq$ constraint.

Step 3. For each constraint without a slack variable s_i, add an artificial variable a_i in the lhs. Also add the sign restriction $a_i \geq 0$.

Step 4. For now, temporarily, ignore the original LP's objective function. Instead solve an LP with an objective function "min $w =$ (sum of all the artificial variables)". This is called the *Phase-I LP*. The act of solving the Phase-I LP will force the artificial variables to be zero.
Because each $a_i \geq 0$, solving the Phase-I LP will result in one of the following three cases:

Case 1. The optimal objective value $w_{\min} > 0$:
In this case, the original LP has no feasible solution.

Case 2. The optimal objective value $w_{\min} = 0$ and no artificial variables existing in the optimal Phase-I basis[7]:
In this case, we drop all columns in the optimal Phase-I tableau that correspond to the artificial variables.[8] Then combine the original objective function with the constraints from the modified optimal Phase-I tableau. This yields the *Phase II-LP*. An optimal

[7] Actually, if no artificial variables exist in the Phase-I basis $\mathbf{x_B}$, we certainly have the corresponding objective value $w_{\min} = 0$ due to the row-0 rhs $\mathbf{c_B^T B^{-1} b}$.

[8] All the artificial variables have completed their missions. Actually as Big-M method when an artificial variable leaves the basis, its column can be immediately dropped from future tableaux during the Simplex procedure. If you do so, you will not need to do the column elimination at the end for Case 2.

solution to the Phase-II LP is an optimal solution to the original LP.

Case 3. The optimal objective value $w_{\min} = 0$ and at least one artificial variable remaining in the optimal Phase-I basis[9]:
In this case, we can find an optimal solution to the original LP by dropping from the optimal Phase-I tableau the columns corresponding to the nonbasic artificial variables and the non-artificial variables that have negative coefficients in row 0 (i.e. by eliminating all the columns corresponding to nonbasic artificial variables[10] or having negative reduced costs; in other words, deleting all the columns corresponding to nonbasic variables which are artificial variables or have negative reduced costs), and reintroducing the original objective function to form the Phase II-LP.

In Phase I, we deal with a relaxed problem where the artificial variables are allowed to be "≥ 0" instead of requiring them to be 0. We will find it easy to get a bfs of the relaxed problem. But what we actually want is to get a bfs of the original LP. In any feasible solution of the Phase-I relaxed problem, we have the objective function value $w \geq 0$. If we can make $w = 0$, the artificial variables will be 0 and the solution will be feasible for the original LP. So in Phase I we have the objective to minimise w.

An interesting point is "Why can we eliminate all the columns corresponding to the non-artificial variables with negative reduced costs in Case 3?". Then "why not do the same elimination in Case 2 as well?" The explanation will be given in the following examples. Now we consider the first example in the previous section.

Example 1

$$\begin{array}{rrcrcr} \min z = & 2x_1 & + & 3x_2 & & \\ s.t. & \frac{1}{2}x_1 & + & \frac{1}{4}x_2 & \leq & 4 \\ & x_1 & + & 3x_2 & \geq & 20 \\ & x_1 & + & x_2 & = & 10 \\ & x_1, x_2 \geq 0 & & & & \end{array}$$

[9]But the artificial variable(s) remaining in the Phase-I basis are certainly equal to zero(s) since $w =$ (sum of all the artificial variables).

[10]Again, when an artificial variable leaves the basis, its column can be immediately dropped. The basic artificial variable(s) cannot be removed since they will be part of the initial basis in the Phase II-LP, i.e. they have not accomplished their missions yet.

As in the Big-M method, Steps 1–3 transform the constraints into

$$\begin{array}{rcrcrcrcrcrcl} \frac{1}{2}x_1 & + & \frac{1}{4}x_2 & + & s_1 & & & & & & & = & 4 \\ x_1 & + & 3x_2 & & & - & e_2 & + & a_2 & & & = & 20 \\ x_1 & + & x_2 & & & & & & & + & a_3 & = & 10 \\ \multicolumn{13}{l}{x_1, x_2, s_1, e_2, a_2, a_3 \geq 0.} \end{array}$$

Then Step 4 yields the following Phase-I LP:

$$\begin{array}{llcrcrcrcrcrcl} \min w = & & & & & & & & & a_2 & + & a_3 & & \\ s.t. & \frac{1}{2}x_1 & + & \frac{1}{4}x_2 & + & s_1 & & & & & & & = & 4 \\ & x_1 & + & 3x_2 & & & - & e_2 & + & a_2 & & & = & 20 \\ & x_1 & + & x_2 & & & & & & & + & a_3 & = & 10 \\ & \multicolumn{13}{l}{x_1, x_2, s_1, e_2, a_2, a_3 \geq 0.} \end{array}$$

We have an obvious initial bfs ($s_1 = 4, a_2 = 20, a_3 = 10$) for Phase I and thus can generate the initial tableau.

basis	x_1	x_2	s_1	e_2	a_2	a_3	rhs
w	0	0	0	0	-1	-1	0
s_1	$\frac{1}{2}$	$\frac{1}{4}$	1	0	0	0	4
a_2	1	3	0	-1	1	0	20
a_3	1	1	0	0	0	1	10

Note, however, that we still need to eliminate a_2 and a_3 in row 0 to satisfy the canonical form. So the first step in Phase I is always to get rid of the coefficients -1 in row 0 (above the artificial variables) by adding all rows corresponding to artificial variables, which are row 2 and row 3 in this case, to row 0. Then We obtain the following new tableau.

basis	x_1	x_2	s_1	e_2	a_2	a_3	rhs	
w	2	4	0	-1	0	0	30	$R_0' \leftarrow R_0 + R_2 + R_3$
s_1	$\frac{1}{2}$	$\frac{1}{4}$	1	0	0	0	4	
a_2	1	3	0	-1	1	0	20	
a_3	1	1	0	0	0	1	10	

The obtained tableau satisfies the conditions: There exists the identity matrix with all corresponding reduced costs being zeros, and the rhs vector is nonnegative.
Now we can proceed with Simplex procedure. Since Phase-I LP is always a minimisation problem, we choose the entering variable by looking for the nonbasic variable with the most positive reduced cost. So x_2 will be entered into the new basis. By the ratio test, the second component of the basic, a_2, is leaving.

basis	x_1	x_2	s_1	e_2	a_2	a_3	rhs	ratio test
w	2	4	0	-1	0	0	30	
s_1	$\frac{1}{2}$	$\frac{1}{4}$	1	0	0	0	4	$\frac{4}{(\frac{1}{4})} = 16$
a_2	1	$\boxed{3}$	0	-1	1	0	20	$\frac{20}{3}*$
a_3	1	1	0	0	0	1	10	$\frac{10}{1} = 10$

Then the pivoting process leads to the following new tableau.

basis	x_1	x_2	s_1	e_2	a_2	a_3	rhs	
w	$\frac{2}{3}$	0	0	$\frac{1}{3}$	-$\frac{4}{3}$	0	$\frac{10}{3}$	$R_0'' \leftarrow R_0' - 4R_2''$
s_1	$\frac{5}{12}$	0	1	$\frac{1}{12}$	-$\frac{1}{12}$	0	$\frac{7}{3}$	$R_1'' \leftarrow R_1' - \frac{1}{4}R_2''$
x_2	$\frac{1}{3}$	1	0	-$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{20}{3}$	$R_2'' \leftarrow \frac{1}{3}R_2'$
a_3	$\frac{2}{3}$	0	0	$\frac{1}{3}$	-$\frac{1}{3}$	1	$\frac{10}{3}$	$R_3'' \leftarrow R_3' - R_2''$

There still exist some positive reduced costs in the current tableau, so the Simplex procedure must be continued.

basis	x_1	x_2	s_1	e_2	a_2	a_3	rhs	ratio test
w	$\frac{2}{3}$	0	0	$\frac{1}{3}$	-$\frac{4}{3}$	0	$\frac{10}{3}$	
s_1	$\frac{5}{12}$	0	1	$\frac{1}{12}$	-$\frac{1}{12}$	0	$\frac{7}{3}$	$\frac{28}{5}$
x_2	$\frac{1}{3}$	1	0	-$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{20}{3}$	20
a_3	$\boxed{\frac{2}{3}}$	0	0	$\frac{1}{3}$	-$\frac{1}{3}$	1	$\frac{10}{3}$	5*

Again, we need to enter x_1 and leave a_3, and do the pivoting. Then the following Simplex tableau is yielded.

basis	x_1	x_2	s_1	e_2	a_2	a_3	rhs
w	0	0	0	0	-1	-1	0
s_1	0	0	1	$-\frac{1}{8}$	$\frac{1}{8}$	$-\frac{5}{8}$	$\frac{1}{4}$
x_2	0	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	5
x_1	1	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$	5

Now all reduced costs are nonpositive, so it is the final optimal Phase-I tableau. Since we have $w_{\min} = 0$ and the optimal basis $(s_1, x_2, x_1) = (\frac{1}{4}, 5, 5)$, i.e. no artificial variables are in the optimal Phase-I basis, the problem is an example of Case 2. We now drop the columns for the artificial variables a_2 and a_3 (we no longer need them) and reintroduce the original objective "min $z = 2x_1 + 3x_2$" to generate the Phase-II tableau as follows.

basis	x_1	x_2	s_1	e_2	rhs
z	-2	-3	0	0	0
s_1	0	0	1	$-\frac{1}{8}$	$\frac{1}{4}$
x_2	0	1	0	$-\frac{1}{2}$	5
x_1	1	0	0	$\frac{1}{2}$	5

Again, before performing Simplex procedure we need to do some EROs for row 0 to make the canonical form as follows.

basis	x_1	x_2	s_1	e_2	rhs	
z	0	0	0	$-\frac{1}{2}$	25	$R_0' \leftarrow R_0 + 3R_2 + 2R_3$
s_1	0	0	1	$-\frac{1}{8}$	$\frac{1}{4}$	
x_2	0	1	0	$-\frac{1}{2}$	5	
x_1	1	0	0	$\frac{1}{2}$	5	

Since none of the reduced costs is positive, this Simplex tableau is the final one. So the optimal solution of the original LP in standard form is

$$(x_1, x_2, s_1, e_2) = (5, 5, \frac{1}{4}, 0)$$

with $z_{\min} = 25$.

Example 2

$$\begin{array}{rrcl} \max\ z = & 4x_1 + 5x_2 & & \\ s.t. & 2x_1 + 3x_2 & \leq & 6 \\ & 3x_1 + x_2 & \geq & 3 \\ & x_1,\ x_2 & \geq & 0 \end{array}$$

We first convert the general form to standard form[11]:

$$\begin{array}{rlcl} \max\ z = & 4x_1 + 5x_2 & & \\ s.t. & 2x_1 + 3x_2 + x_3 & = & 6 \\ & 3x_1 + x_2 \qquad - x_4 & = & 3 \\ & x_1,\ x_2,\ x_3,\ x_4 & \geq & 0 \end{array}$$

Then for the second constraint, we need to introduce an artificial variable a_1 since x_4 cannot be a part of the initial bfs.
The Phase-I LP is shown as follows:

$$\begin{array}{rlcl} \min\ w = & \qquad\qquad\qquad a_1 & & \\ s.t. & 2x_1 + 3x_2 + x_3 & = & 6 \\ & 3x_1 + x_2 \qquad - x_4 + a_1 & = & 3 \\ & x_1, x_2,\ x_3, x_4,\ a_1 & \geq & 0 \end{array}$$

Choose $(x_3, a_1) = (6, 3)$ as the initial basis and draw up the following tableau:

basis	x_1	x_2	x_3	x_4	a_1	rhs
w	0	0	0	0	-1	0
x_3	2	3	1	0	0	6
a_1	3	1	0	-1	1	3

Get rid of the coefficients -1 in row 0 above a_1 by adding row 2 to row 0.

[11]Notice that the notation used for slack or excess variables is flexible.

basis	x_1	x_2	x_3	x_4	a_1	rhs	
w	3	1	0	-1	0	3	$R_0' \leftarrow R_0 + R_2$
x_3	2	3	1	0	0	6	
a_1	3	1	0	-1	1	3	

Now we can progress with the Simplex method. Choose the most positive reduced cost to determine the entering variable, and find the leaving variable using the ratio test as follows.

basis	x_1	x_2	x_3	x_4	a_1	rhs	ratio test
w	3	1	0	-1	0	3	
x_3	2	3	1	0	0	6	$\frac{6}{2} = 3$
a_1	$\boxed{3}$	1	0	-1	1	3	$\frac{3}{3} = 1$*

Entering x_1 and leaving a_1, we do the pivoting and have the following new tableau.

basis	x_1	x_2	x_3	x_4	a_1	rhs
w	0	0	0	0	-1	0
x_3	0	$\frac{7}{3}$	1	$\frac{2}{3}$	-$\frac{2}{3}$	4
x_1	1	$\frac{1}{3}$	0	-$\frac{1}{3}$	$\frac{1}{3}$	1

Since no positive reduced cost exists, this is the final tableau. The optimal objective value is $w_{\min} = 0$, and a_1 is not in the optimal Phase-I basis. This is another example of Case 2.[12] Hence we can generate Phase-II initial

[12]Now we can discuss the question "Why not eliminate all the columns corresponding to the non-artificial variables with negative reduced costs in Case 2?". Recall that only nonbasic variables have non-zero reduced costs. Notice that in the previous Example 1 (also of Case 2) the only nonbasic non-artificial variable in the optimal Phase-I tableau is e_2 whose reduced cost is 0. In Example 2, the nonbasic non-artificial variables are x_2 and x_4 whose reduced costs are both 0 again. Actually, this is not a coincidence. In Case 2, each nonbasic non-artificial variable x_j has a reduced cost $\mathbf{c}^T_{\mathbf{B}}\mathbf{B}^{-1}\mathbf{A}_j - c_j = 0$ since $\mathbf{c_B} = \mathbf{0}$ and $c_j = 0$ in the Phase-I LP of standard form (i.e. no non-artificial variable exists in the Phase-I objective function

tableau by combining the original objective function with the constraints from the optimal Phase-I tableau. In other words, the Phase-I optimal basis $(x_3, x_1) = (4, 1)$ is exactly the initial feasible basis of the Phase-II LP.

Dropping the column of artificial variable a_1 and reintroducing the original objective "max $z = 4x_1 + 5x_2$", we have the Phase-II tableau as follows.

basis	x_1	x_2	x_3	x_4	rhs
z	-4	-5	0	0	0
x_3	0	$\frac{7}{3}$	1	$\frac{2}{3}$	4
x_1	1	$\frac{1}{3}$	0	-$\frac{1}{3}$	1

Again, some EROs in row 0 are needed to make the canonical form.

basis	x_1	x_2	x_3	x_4	rhs	
z	0	-$\frac{11}{3}$	0	-$\frac{4}{3}$	4	$R_0' \leftarrow R_0 + 4R_2$
x_3	0	$\frac{7}{3}$	1	$\frac{2}{3}$	4	
x_1	1	$\frac{1}{3}$	0	-$\frac{1}{3}$	1	

Now we can proceed with the Simplex method. Choose the most negative reduced cost to determine the entering variable, and find the leaving variable using the ratio test.

basis	x_1	x_2	x_3	x_4	rhs	ratio test
z	0	-$\frac{11}{3}$	0	-$\frac{4}{3}$	4	
x_3	0	$\boxed{\frac{7}{3}}$	1	$\frac{2}{3}$	4	$\frac{4}{(\frac{7}{3})} = \frac{12}{7}*$
x_1	1	$\frac{1}{3}$	0	-$\frac{1}{3}$	1	$\frac{1}{(\frac{1}{3})} = 3$

Entering x_2 and leaving x_3, we do the pivoting and have the following new tableau.

basis	x_1	x_2	x_3	x_4	rhs	
z	0	0	$\frac{11}{7}$	-$\frac{2}{7}$	$\frac{72}{7}$	$R_0'' \leftarrow R_0' + \frac{11}{3}R_1''$
x_2	0	1	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{12}{7}$	$R_1'' \leftarrow \frac{3}{7}R_1'$ (go first)
x_1	1	0	-$\frac{1}{7}$	-$\frac{3}{7}$	$\frac{3}{7}$	$R_2'' \leftarrow R_2' - \frac{1}{3}R_1''$

Another Simplex iteration starts with

basis	x_1	x_2	x_3	x_4	rhs	ratio test
z	0	0	$\frac{11}{7}$	-$\frac{2}{7}$	$\frac{72}{7}$	
x_2	0	1	$\frac{3}{7}$	$\boxed{\frac{2}{7}}$	$\frac{12}{7}$	$\frac{(\frac{12}{7})}{\frac{2}{7}} = 6*$
x_1	0	0	-$\frac{1}{7}$	-$\frac{3}{7}$	$\frac{3}{7}$	null

Entering x_4 and leaving x_2, we do the pivoting and have the following new tableau.

basis	x_1	x_2	x_3	x_4	rhs	
z	0	1	2	0	12	$R_0''' \leftarrow R_0'' + R_1''$
x_4	0	$\frac{7}{2}$	$\frac{3}{2}$	1	6	$R_1''' \leftarrow \frac{7}{2}R_1''$ (go first)
x_1	1	$\frac{3}{2}$	$\frac{1}{2}$	0	3	$R_2''' \leftarrow R_2'' + \frac{3}{7}R_1'''$

Since none of the reduced costs is negative, this is the final Phase-II tableau.

basis	x_2	x_3	x_4	x_5	x_6	a_1	a_2	rhs
z	4	0	7	0	0	0	0	7
a_1	-1	0	0	2	0	1	0	0
a_2	1	0	0	-2	0	0	1	0
x_3	1	1	$\frac{1}{2}$	$\frac{3}{2}$	0	0	0	$\frac{7}{2}$
x_6	1	0	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$

Since all reduced costs are nonnegative, this is the optimal Phase-II tableau. The optimal solution to the original LP is

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, \frac{7}{2}, 0, 0, \frac{1}{2})$$

with the optimal objective value $z_{\max} = 7$.

Further reading: Section 4.12 an 4.13 in the reference book "Operations Research: Applications and Algorithms" (Winston, 2004)

2. The Revised Simplex Method

2.1 Introduction

This chapter introduces the revised Simplex method, a modified implementation of the Simplex method in order to improve the computational efficiency, which is crucial for large-scale LP models. Instead of generating a full Simplex tableau in each iteration, the revised Simplex algorithm reduces computation by utilising a simplified algebraic Simplex tableau and formulae to update the bfs.

Large-scale LP problems contain large amounts of data, and require even larger numbers of computations in each Simplex iteration. Actually, some data generated in the Simplex tableau in each iteration are irrelevant. In particular, the only nonbasic variable of interest in each iteration is the one that has to enter the basis. Similarly, a Simplex tableau with many columns and few rows requires much more updating of nonbasic columns, which may be unnecessary until many iterations later, if ever. The revised Simplex method substantially reduces avoidable data storage and computational effort while the underlying Simplex logic remains the same.

Recall that the Simplex procedure in tabular form is actually based upon the Simplex algebraic formulae.[1] Given any basis $\mathbf{x_B}$, we can generate its Simplex tableau using the algebraic Simplex tableau. Again, recall that for an LP in standard form

$$\begin{aligned} \min\,(\text{or max})\ z = \ & \mathbf{c}^T\mathbf{x} \\ s.t. \qquad\quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

the algebraic Simplex tableau for each iteration with a chosen basis $\mathbf{x_B}$ is shown below.

basis	$\mathbf{x}$	rhs
z	$\mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}^T$	$\mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{b}$
$\mathbf{x_B}$	$\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{B}^{-1}\mathbf{b}$

Since a basis $\mathbf{x_B}$ is chosen, its corresponding basic matrix $\mathbf{B}$ and objective coefficient vector $\mathbf{c_B}$ can be obtained by inspection from $\mathbf{A}$ and $\mathbf{c}$, respectively, in the LP of the standard form. After calculating $\mathbf{B}^{-1}$, we can compute any portion of the Simplex tableau. This means that if a computer is programmed to perform the Simplex algorithm, all the computer needs to store for any iteration are the original LP in standard form, the chosen basis and its corresponding $\mathbf{B}^{-1}$. In other words, the key part of the Simplex procedure after choosing a basis $\mathbf{x_B}$ is the calculation of $\mathbf{B}^{-1}$.[2] If

[1] Please refer to Lecture Note – Part 3.

[2] Although we perform the pivoting (i.e. EROs) rather than the calculation of $\mathbf{B}^{-1}$ in the tabular Simplex procedure, we later will see the equivalence between them.

for each iteration we calculate the current $\mathbf{B}^{-1}$ from scratch, it would take a lot of work and be inefficient. The key idea of the revised Simplex method stems from how to efficiently keep track of the changes in $\mathbf{B}^{-1}$, i.e. efficiently update $\mathbf{B}^{-1}$. Later we will show that $\mathbf{B}^{-1}$ can actually be obtained in the row-operated constraint matrix in the Simplex tableau. Instead of producing the full Simplex tableau, the revised Simplex method adopts a simplified Simplex tableau to update $\mathbf{B}^{-1}$ and some essential data.

Recall that we also can generate the algebraic Simplex tableau, where the decision variable vector $\mathbf{x}$ is decomposed into the basis $\mathbf{x_B}$ and nonbasis $\mathbf{x_N}$, i.e. $\mathbf{x}^T = (\mathbf{x}_\mathbf{N}^T \,|\, \mathbf{x}_\mathbf{B}^T)$, $\mathbf{c}^T = (\mathbf{c}_\mathbf{N}^T \,|\, \mathbf{c}_\mathbf{B}^T)$ and $\mathbf{A} = (\mathbf{N} \,|\, \mathbf{B})$, as shown below.

basis	$\mathbf{x_N}$	$\mathbf{x_B}$	rhs
z	$\mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_\mathbf{N}^T$	$\mathbf{0}^T$	$\mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{b}$
$\mathbf{x_B}$	$\mathbf{B}^{-1}\mathbf{N}$	$\mathbf{I}$	$\mathbf{B}^{-1}\mathbf{b}$

Note that the columns corresponding to $\mathbf{x_B}$ make the yielded Simplex tableau be in canonical form. Recalling how an 'initial' basis is always generated in the system of equations $\mathbf{Ax} = \mathbf{b}$ satisfying the canonical form[3], we notice that the columns of $\mathbf{A}$ associated with an initial basis $\mathbf{x}_{\mathbf{B}_0}$ simply constitute the identity matrix, i.e. $\mathbf{B}_0 = \mathbf{I}$ and $\mathbf{A} = (\mathbf{N}_0 \,|\, \mathbf{I})$. Then the initial Simplex tableau can be shown below.

[3] After introducing the Big-M method, we can assume without loss of generality that an initial basis can always be produced for an LP in standard form.

basis	$\mathbf{x}_{\mathbf{N}_0}$	$\mathbf{x}_{\mathbf{B}_0}$	rhs
z	$\mathbf{c}_{\mathbf{B}_0}^T\mathbf{N}_0 - \mathbf{c}_{\mathbf{N}_0}^T$	$\mathbf{0}^T$	$\mathbf{c}_{\mathbf{B}_0}^T\mathbf{b}$
$\mathbf{x}_{\mathbf{B}_0}$	$\mathbf{N}_0$	$\mathbf{I}$	$\mathbf{b}$

Assume now that we determine a new (adjacent) basis $\mathbf{x}_{\mathbf{B}_1}$ with the basic matrix $\mathbf{B}_1$ and perform the Simplex procedure (i.e. pivoting) to produce the corresponding new Simplex tableau from this initial one. The algebraic Simplex tableau shows that in the new tableau the rhs of $\mathbf{x}_{\mathbf{B}_1}$ is $\mathbf{B}_1^{-1}\mathbf{b}$. Therefore, the obtained constraint matrix of the new tableau is exactly the constraint matrix of the initial tableau multiplied by $\mathbf{B}_1^{-1}$, i.e.

$$\mathbf{B}_1^{-1}(\mathbf{N}_0 \,|\, \mathbf{I}) = (\mathbf{B}_1^{-1}\mathbf{N}_0 \,|\, \mathbf{B}_1^{-1}\mathbf{I}) = (\mathbf{B}_1^{-1}\mathbf{N}_0 \,|\, \mathbf{B}_1^{-1}).$$

Actually, the performed EROs in pivoting completes the linear transformations on the system of constraints/equations. These transformations are recorded in some part of the new tableau, which originally was $\mathbf{I}$ in the initial tableau. It is exactly $\mathbf{B}_1^{-1}$, which is a record of all the Simplex EROs conducted from the initial tableau to the new one.

Then the new reduced cost is

$$\begin{aligned}
\mathbf{c}_{\mathbf{B}_1}^T\mathbf{B}_1^{-1}\mathbf{A} - \mathbf{c}^T &= \mathbf{c}_{\mathbf{B}_1}^T\mathbf{B}_1^{-1}(\mathbf{N}_0 \,|\, \mathbf{I}) - (\mathbf{c}_{\mathbf{N}_0}^T \,|\, \mathbf{c}_{\mathbf{B}_0}^T) \\
&= (\mathbf{c}_{\mathbf{B}_1}^T\mathbf{B}_1^{-1}\mathbf{N}_0 \,|\, \mathbf{c}_{\mathbf{B}_1}^T\mathbf{B}_1^{-1}) - (\mathbf{c}_{\mathbf{N}_0}^T \,|\, \mathbf{c}_{\mathbf{B}_0}^T) \\
&= (\mathbf{c}_{\mathbf{B}_1}^T\mathbf{B}_1^{-1}\mathbf{N}_0 - \mathbf{c}_{\mathbf{N}_0}^T \,|\, \mathbf{c}_{\mathbf{B}_1}^T\mathbf{B}_1^{-1} - \mathbf{c}_{\mathbf{B}_0}^T).
\end{aligned}$$

Therefore, the new Simplex tableau after the first iteration is shown below.

basis	$\mathbf{x}_{\mathbf{N}_0}$	$\mathbf{x}_{\mathbf{B}_0}$	rhs
z	$\mathbf{c}_{\mathbf{B}_1}^T\mathbf{B}_1^{-1}\mathbf{N}_0 - \mathbf{c}_{\mathbf{N}_0}^T$	$\mathbf{c}_{\mathbf{B}_1}^T\mathbf{B}_1^{-1} - \mathbf{c}_{\mathbf{B}_0}^T$	$\mathbf{c}_{\mathbf{B}_1}^T\mathbf{B}_1^{-1}\mathbf{b}$
$\mathbf{x}_{\mathbf{B}_1}$	$\mathbf{B}_1^{-1}\mathbf{N}_0$	$\mathbf{B}_1^{-1}$	$\mathbf{B}_1^{-1}\mathbf{b}$

From the above tableau[4] we can observe that

- $\mathbf{B}_1^{-1}$ is the constraint submatrix whose columns are associated with the 'original/initial' basis $\mathbf{x}_{\mathbf{B}_0}$;
- Once $\mathbf{B}_1^{-1}$ is known, we can calculate all other parts in the new tableau;
- It will be convenient to calculate all values in row 0 if we keep track of the value $\mathbf{c}_{\mathbf{B}_1}^T\mathbf{B}_1^{-1}$.

Assume that $(\mathbf{x}_{\mathbf{N}_1}^T \,|\, \mathbf{x}_{\mathbf{B}_1}^T)$ is not optimal. The Simplex procedure keeps generating a new (adjacent) basis $\mathbf{x}_{\mathbf{B}_2}$ for the second iteration. Then it can be easily shown that the following new Simplex tableau will be yielded.

[4] If the objective function $z = \mathbf{c}^T\mathbf{x} = \mathbf{c}_{\mathbf{N}_0}^T\mathbf{x}_{\mathbf{N}_0} + \mathbf{c}_{\mathbf{B}_0}^T\mathbf{x}_{\mathbf{B}_0}$ naturally abides by the canonical form for the initial basis $\mathbf{x}_{\mathbf{B}_0}$, i.e. $z = \mathbf{c}_{\mathbf{N}_0}^T\mathbf{x}_{\mathbf{N}_0}$ and $\mathbf{c}_{\mathbf{B}_0} = \mathbf{0}$, then the above Simplex tableau can be shown as follows.

basis	$\mathbf{x}_{\mathbf{N}_0}$	$\mathbf{x}_{\mathbf{B}_0}$	rhs
z	$\mathbf{c}_{\mathbf{B}_1}^T\mathbf{B}_1^{-1}\mathbf{N}_0 - \mathbf{c}_{\mathbf{N}_0}^T$	$\mathbf{c}_{\mathbf{B}_1}^T\mathbf{B}_1^{-1}$	$\mathbf{c}_{\mathbf{B}_1}^T\mathbf{B}_1^{-1}\mathbf{b}$
$\mathbf{x}_{\mathbf{B}_1}$	$\mathbf{B}_1^{-1}\mathbf{N}_0$	$\mathbf{B}_1^{-1}$	$\mathbf{B}_1^{-1}\mathbf{b}$

A common example of the assumption is exactly the LP with all the constraints being '$\leq$'.

basis	$\mathbf{x}_{\mathbf{N}_0}$	$\mathbf{x}_{\mathbf{B}_0}$	rhs
z	$\mathbf{c}_{\mathbf{B}_2}^T\mathbf{B}_2^{-1}\mathbf{N}_0 - \mathbf{c}_{\mathbf{N}_0}^T$	$\mathbf{c}_{\mathbf{B}_2}^T\mathbf{B}_2^{-1} - \mathbf{c}_{\mathbf{B}_0}^T$	$\mathbf{c}_{\mathbf{B}_2}^T\mathbf{B}_2^{-1}\mathbf{b}$
$\mathbf{x}_{\mathbf{B}_2}$	$\mathbf{B}_2^{-1}\mathbf{N}_0$	$\mathbf{B}_2^{-1}$	$\mathbf{B}_2^{-1}\mathbf{b}$

Therefore, once a new basis $\mathbf{x}_\mathbf{B}$ is determined for any iteration, we can update $\mathbf{B}^{-1}$ using EROs for a 'partial' instead of the 'entire' bottom section of the Simplex tableau. This leads to the revised Simplex algorithm, which updates $\mathbf{B}^{-1}$ between iterations via a simplified tableau.