Example 2

Second example to solve:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \qquad \left[\begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

To bring this to EF we eliminate x_1 in equation 3:

$$\text{Eq3} + 4 * \text{Eq1} \to \text{Eq3} \qquad \text{or} \qquad \text{R3} + 4 * \text{R1} \to \text{R3}$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & & -3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \left[\begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

Example 2

$$\begin{array}{rcrcrcl} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & & -3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \left[\begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

Next, we eliminate x_2 in equation 3. But first, factor R2.

$$\text{Eq2} \to \frac{1}{2} \times \text{Eq2} \qquad \text{or} \qquad \text{R2} \to \frac{1}{2} \times \text{R2}$$

$$\begin{array}{rcrcrcl} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & -3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \left[\begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

Next, $\text{Eq3} + 3 * \text{Eq2} \to \text{Eq3}$ or $\text{R3} + 3 * \text{R2} \to \text{R3}$

$$\begin{array}{rcrcrcl} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Example 2

$$\begin{array}{rcrcrcl} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

The matrix of the system is now in an echelon form.

Now we continue towards REF, or we can also solve it directly:

Solving directly:

$$\begin{array}{lcl} \text{Eq3} & \Rightarrow & x_3 = 3 \\ \text{Eq2} & \Rightarrow & x_2 = 4x_3 + 4 = 4 \times 3 + 4 = 16 \\ \text{Eq1} & \Rightarrow & x_1 = 2x_2 - x_3 + 0 = 2 \times 16 - 3 = 29 \end{array}$$

This is called **backward substitution**.

Example 2

Continue with the elimination into REF

$$\begin{array}{rcrcrcl} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Elimination above the pivots (yes, may be done also in this order):

$$\mathsf{Eq1} + 2 * \mathsf{Eq2} \to \mathsf{Eq1} \qquad \text{or} \qquad \mathsf{R1} + 2 * \mathsf{R2} \to \mathsf{R1}$$

$$\begin{array}{rcrcrcl} x_1 & & & - & 7x_3 & = & 8 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{cccc} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Example 2

$$\begin{array}{rrrcl} x_1 & & - \; 7x_3 & = & 8 \\ & x_2 & - \; 4x_3 & = & 4 \\ & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{cccc} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$\mathsf{Eq1} + 7 * \mathsf{Eq3} \to \mathsf{Eq1}$ or $\mathsf{R1} + 7 * \mathsf{R2} \to \mathsf{R1}$

$\mathsf{Eq2} + 4 * \mathsf{Eq3} \to \mathsf{Eq2}$ or $\mathsf{R2} + 4 * \mathsf{R3} \to \mathsf{R2}$

$$\begin{array}{rrrcl} x_1 & & & = & 29 \\ & x_2 & & = & 16 \\ & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{cccc} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Example with no solutions

Third example

$$\begin{array}{rcrcrcl} & & x_2 & - & 4x_3 & = & 8 \\ 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ 5x_1 & - & 8x_2 & + & 7x_3 & = & 1 \end{array} \qquad \left[\begin{array}{cccc} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{array}\right]$$

Upon doing $\mathrm{R}_1 \leftrightarrow \mathrm{R}_2$ and $\mathrm{R}_3 \to (2\mathrm{R}_3 - 5\mathrm{R}_2) + \mathrm{R}_1$ we get

$$\begin{array}{rcrcrcl} 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ & & x_2 & - & 4x_3 & = & 8 \\ & & & & 0 & = & 5 \end{array} \qquad \left[\begin{array}{cccc} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

The **inconsistency** $0 = 5$ implies that
this system does not have any solutions.

Example with infinitely many solutions

Fourth example

$$\begin{array}{rcrcrcr} & & x_2 & - & 4x_3 & = & 6 \\ 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ 5x_1 & - & 8x_2 & + & 7x_3 & = & -1/2 \end{array} \qquad \left[\begin{array}{cccc} 0 & 1 & -4 & 6 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & -1/2 \end{array}\right]$$

Doing again $\mathsf{R}_1 \leftrightarrow \mathsf{R}_2$ and $\mathsf{R}_3 \to (2\mathsf{R}_3 - 5\mathsf{R}_2) + \mathsf{R}_1$ yields

$$\begin{array}{rcrcrcr} 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ & & x_2 & - & 4x_3 & = & 6 \\ & & & & 0 & = & 0 \end{array} \qquad \left[\begin{array}{cccc} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

3 equations in 3 unknowns $\longrightarrow$ 2 equations in 3 unknowns

$\Rightarrow$ only two independent equations

No contradiction, but no unique solution (infinitely many)

$\mathbf{A}^{-1}$ by Gaussian elimination

Example: Find inverse of a matrix, if it exists

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$$

We form the extended augmented matrix

$$\begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$

$\mathsf{R}_1 \leftrightarrow \mathsf{R}_2$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$

$\mathsf{R}_3 \to \mathsf{R}_3 - 4\mathsf{R}_1$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{bmatrix}$$

$\mathsf{R}_3 \to \mathsf{R}_3 + 3\mathsf{R}_2$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix}$$

$\mathsf{R}_3 \to \mathsf{R}_3/2$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

$\mathsf{R}_2 \to \mathsf{R}_2 - 2\mathsf{R}_3$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

$\mathsf{R}_1 \to \mathsf{R}_1 - 3\mathsf{R}_3$

$$\begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

Thus $\mathbf{A}$ is invertible and $\mathbf{A}^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$