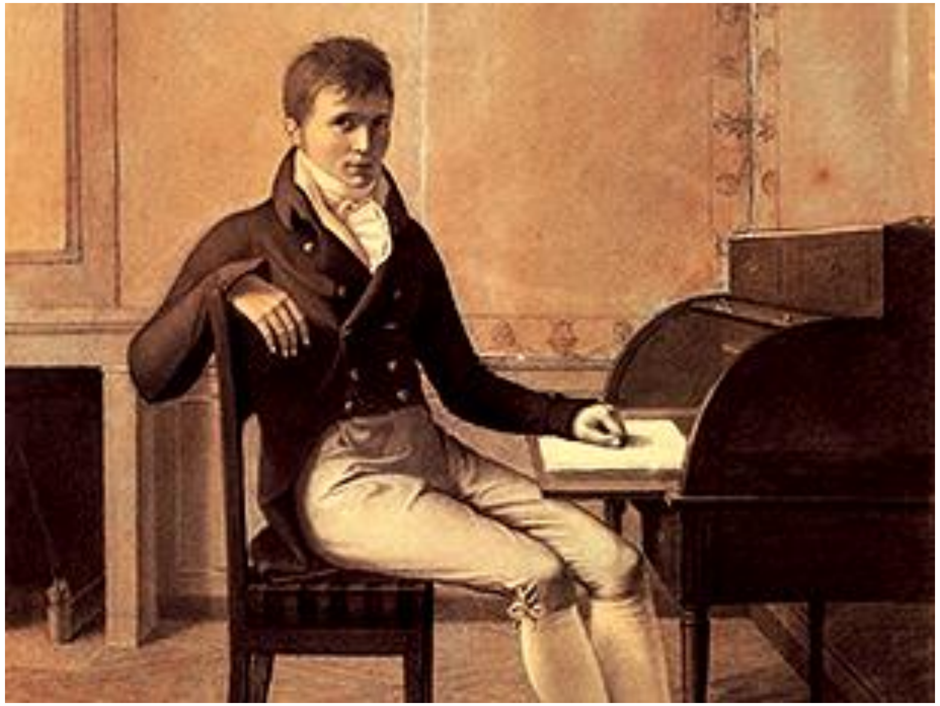
37161 Probability and Random Variables

Lecture 7

Poisson Variables

- If we have a series of events which occur independently of each other in space or time, but at some known average rate, we have a **Poisson** variable.

- For example, if we assume that in a large building, a lightbulb needs replacing on average once per hour then (assuming bulbs don't break with any pattern i.e. no power surge etc.) we would model the number N of bulbs needing replacement in a given hour by $N \sim Poi(1)$.



Siméon-Denis Poisson

(*1781 – 1840*)

Poisson Variables

- The Poisson variable requires just one parameter – λ – the average rate at which the event occurs.

- As it is simply a count, the range of $N \sim Poi(\lambda)$ is $\{0,1,2,3,...\}$

- The probability mass function of N is $P(N=k) = \begin{cases} \dfrac{e^{-\lambda}\lambda^k}{k!} & k \in \{0,1,2,3,...\} \\ 0 & \text{otherwise} \end{cases}$

Poisson Variables

- We can verify that this is a valid probability mass function since

$$\sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^{k}}{k!} = e^{-\lambda}\left[1 + \frac{\lambda}{1!} + \frac{\lambda}{2!}^{2} + \frac{\lambda^{3}}{3!} + \ldots\right] = e^{-\lambda}\left[e^{\lambda}\right] = 1$$

- The expectation of this is $E(N) = \sum_{k=0}^{\infty} k \frac{e^{-\lambda}\lambda^{k}}{k!} = \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^{k}}{(k-1)!} = \sum_{k=0}^{\infty} \frac{\lambda e^{-\lambda}\lambda^{(k-1)}}{(k-1)!}$

$$= \lambda e^{-\lambda}\left[1 + \frac{\lambda}{1!} + \frac{\lambda}{2!}^{2} + \frac{\lambda^{3}}{3!} + \ldots\right] = \lambda$$

Poisson Variables: Example

- Consider modelling the arrival of calls into a call centre.
- It is noted that the centre receives an average of 20 calls per hour throughout its working day, with all calls arriving independently of one another and no two periods (of equal length) of the day more or less likely to receive calls than the other.
- The number of calls in a given hour therefore $\sim Poi(20)$.
- For example, the probability that, during one hour, exactly 17 calls arrive is $\dfrac{e^{-20}20^{17}}{17!} \approx 7.6\%$

Poisson Variables: How Do They Arise?

- The Poisson variable arises from the limit case of a Binomial distribution.

- We can consider a Binomial distribution to be the sum of independent Bernoulli variables, each occurring with probability p.

- Breaking a time period (e.g. one day) into N periods of equal length, we can then regard (for sufficiently large N) the experiment as being a series of N Bernoulli trials, i.e. whether or not a the event occurs during each tiny time period.

Poisson Variables: How Do They Arise?

- We assume that since N is very large, the length of each period is sufficiently short that two such events cannot occur in any one period.

- Instead of taking $X \sim Poi(\lambda)$, we can take $X \sim Bin\left(N, \frac{\lambda}{N}\right)$ for very large N.

- Note that the expectation of both variables is λ.

Poisson Variables: How Do They Arise?

- Since $X \sim Bin\left(N, \frac{\lambda}{N}\right)$ we know that $P(X=k) = \frac{N!}{k!(N-k)!}\left(\frac{\lambda}{N}\right)^k \left(1-\frac{\lambda}{N}\right)^{N-k}$
- $P(X=k) = \frac{N!}{N^k(N-k)!}\frac{\lambda^k}{k!}\left(1-\frac{\lambda}{N}\right)^N \left(1-\frac{\lambda}{N}\right)^{-k}$
- As $N \to \infty$ $\left(1-\frac{\lambda}{N}\right)^{-k} \to 1$ and $\frac{N!}{N^k(N-k)!} = 1\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\left(1-\frac{3}{N}\right)\ldots\left(1-\frac{k-1}{N}\right) \to 1$
- Also $\left(1-\frac{\lambda}{N}\right)^N \to e^{-\lambda}$ as $N \to \infty$
- Hence, we get the result that if $X \sim Poi(\lambda)$, $P(X=k) = \frac{e^{-\lambda}\lambda^k}{k!}$

Merging of Poisson Variables

- Consider now two independent Poisson variables, $X \sim Poi(\lambda)$ and $Y \sim Poi(\mu)$.

- What is the distribution of $Z = X + Y$?

- We calculate the probability that $Z = k$ by summing all the ways this can happen i.e.

$$P(Z = k) = \sum_{j=0}^{k} P(X = j)P(Y = k - j)$$

- For example, the event $Z = 3$ is the union of the events $\{X = 0\} \cap \{Y = 3\}, \{X = 1\} \cap \{Y = 2\},$ $\{X = 2\} \cap \{Y = 1\}$ and $\{X = 3\} \cap \{Y = 0\}$

Merging of Poisson Variables

- $P(Z=k)=\sum_{j=0}^{k}P(X=j)P(Y=k-j) = \sum_{j=0}^{k}\left[\frac{e^{-\lambda}\lambda^{j}}{j!}\right]\left[\frac{e^{-\mu}\mu^{k-j}}{(k-j)!}\right] = e^{-\lambda}e^{-\mu}\sum_{j=0}^{k}\left[\frac{\lambda^{j}}{j!}\right]\left[\frac{\mu^{k-j}}{(k-j)!}\right]$

$$= \frac{e^{-\lambda}e^{-\mu}}{k!}\sum_{j=0}^{k}\frac{k!}{j!(k-j)!}\lambda^{j}\mu^{k-j} = \frac{e^{-\lambda}e^{-\mu}}{k!}\sum_{j=0}^{k}\binom{k}{j}\lambda^{j}\mu^{k-j}$$

- Noting that $(A+B)^{n}=A^{n}+\binom{n}{1}A^{n-1}B+\binom{n}{2}A^{n-2}B^{2}+\ldots+\binom{n}{n-1}AB^{n-1}+B^{n}$, we see that $\sum_{j=0}^{k}\binom{k}{j}\lambda^{j}\mu^{k-j}=(\lambda+\mu)^{k}$
- Hence $P(Z=k)=\sum_{j=0}^{k}P(X=j)P(Y=k-j)=\frac{e^{-(\lambda+\mu)}(\lambda+\mu)^{k}}{k!}$ therefore $Z\sim Poi(\lambda+\mu)$.

Merging of Poisson Variables: Example

- A website models the number of visitors it receives each hour by Poisson variables.
- It is noted that, independently, each hour an average of 10,000 visitors load the site without making a purchase and an average of 2,000 visitors load the site and do make a purchase.
- What is the distribution of the total number of visitors per hour?
- If $X_{non-purchase} \sim Poi(10{,}000)$ and $X_{purchase} \sim Poi(2{,}000)$, then

$$X_{total} = X_{non-purchase} + X_{purchase} \sim Poi(12{,}000)$$

Splitting of Poisson Variables

- Let $X \sim Poi(\lambda)$ and assume that all events counted are classified as either Type 1 or Type 2 such that each event independently is Type 1 with probability *p*.

- What is the distribution of X_1 , the number of Type 1 events which occur?

- In order to calculate $P(X_1 = n)$, we first calculate the conditional probability $P(X_1 = n | X = n+m)$ and then sum over all possible ways $X = n+m$. (i.e. $P(A) = \sum_i P(A|B_i)P(B_i)$ assuming $\sum_i P(B_i) = 1$)
- This makes the calculation simpler since, conditional on knowing $X = n+m,\ X_1 \sim Bin(n+m, p)$.

Splitting of Poisson Variables

- $P(X_1 = n | X = n+m) = \binom{n+m}{n} p^n (1-p)^m$ and $P(X = n+m) = \dfrac{e^{-\lambda}\lambda^{(n+m)}}{(n+m)!}$ hence

$$P(X_1 = n) = \sum_{m=0}^{\infty} \binom{n+m}{n} p^n (1-p)^m \frac{e^{-\lambda}\lambda^{(n+m)}}{(n+m)!} = \sum_{m=0}^{\infty} \frac{(n+m)!}{n!m!} p^n (1-p)^m \frac{e^{-\lambda}\lambda^n \lambda^m}{(n+m)!}$$

$$= \frac{p^n \lambda^n e^{-\lambda p}}{n!} \sum_{m=0}^{\infty} (1-p)^m \frac{e^{-\lambda(1-p)}\lambda^m}{m!}$$

$$= \frac{(\lambda p)^n e^{-\lambda p}}{n!} \sum_{m=0}^{\infty} \frac{e^{-\lambda(1-p)} \left[\lambda(1-p)\right]^m}{m!} = \frac{(\lambda p)^n e^{-\lambda p}}{n!}$$

- That is, $X_1 \sim Poi(\lambda p)$
- So, for example, if a shop expects 12 customers per hour and 50% of these are male then (assuming each arrival is independent), the number of males arrive per hour $\sim Poi(6)$.

Poisson Processes

- As we can merge together or split independent Poisson variables, we can define a Poisson process, $X(t)$ with rate $\lambda > t$ for any time $t \geq 0$ such that:

- $X(t)$ is a counting process with $X(0) = 0$ (i.e. has range $\{0,1,2,3,...\}$) with independent stationary increments i.e. the number of events counted in an interval of length T depends only on T and that any the number of events counted in any two disjoint intervals are independent.
- For any $t \geq 0$, the probability that there are k events counted during an interval of length t is $\frac{e^{-\lambda t}(\lambda t)^k}{k!}$ i.e. $X(t) \sim Poi(\lambda t)$.

Poisson Processes: Waiting Times

- Let $X(t)$ be a Poisson process with rate λ.
- Consider W, the time until the first event,
- Clearly, for $t > 0$, the events $\{X(t) = 0\}$ and $\{W > t\}$ are identical, since having no events occur in a timestep of length t is equivalent to having to wait more than t for another event to occur.
- $P(X(t) = 0) = P(W > t) = \dfrac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-\lambda t}$

Poisson Processes: Waiting Times

- $P(X(t)=0)=P(W>t)=\dfrac{e^{-\lambda t}(\lambda t)^0}{0!}=e^{-\lambda t}$ hence the cumulative density of W is $F_W(t)=P(W\le t)=1-e^{-\lambda t}$

- Differentiating gives the probability density function $f_W(t)=\dfrac{dF_W}{dt}=\lambda e^{-\lambda t}$.

- We say that W is an **exponential** random variable with rate $\lambda>0$ if it has density function

$$f(w)=\begin{cases}\lambda e^{-\lambda w} & w>0\\ 0 & \text{otherwise}\end{cases}.$$

We denote this as $W\sim\exp(\lambda)$

Exponential Distribution

- It is easy to verify that $f(w)$ is a valid density function, since

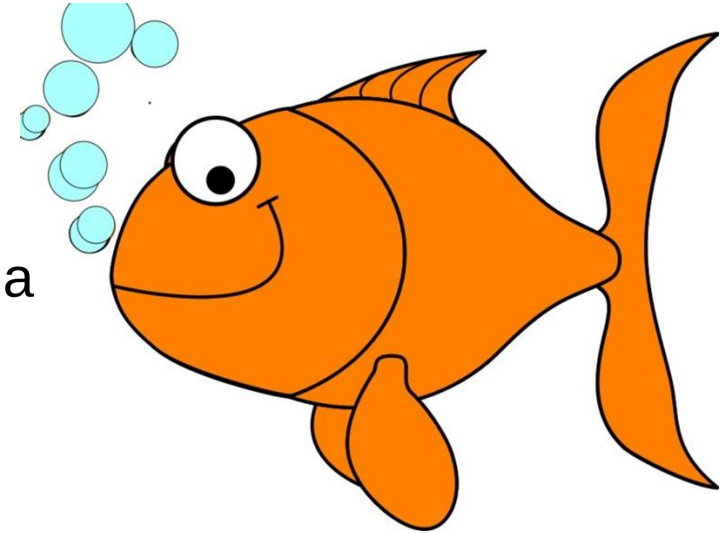
$$\cdot \int_{-\infty}^{\infty} f(w)dw = \int_0^{\infty} \lambda e^{-\lambda w} dw = \left[-e^{-\lambda w}\right]_0^{\infty} = 1$$

- The expectation is obtained via integration by parts

$$\int_{-\infty}^{\infty} wf(w)dw = \int_0^{\infty} \lambda w e^{-\lambda w} dw = \left[-we^{-\lambda w}\right]_0^{\infty} + \int_0^{\infty} e^{-\lambda w} dw = 0 + \left[\frac{-e^{-\lambda w}}{\lambda}\right]_0^{\infty} = \frac{1}{\lambda}$$

- $E(W) = \frac{1}{\lambda}$ is perhaps intuitive, since if, for example, we expect 5 arrivals per hour, then the expected time until an additional arrival is 0.2 hours.

Exponential Variables: No Memory Property



- Let $W \sim \exp(\lambda)$ and consider two times $a, b > 0$.
- What is $P(W > a + b \mid W > a)$ i.e. the chance that we need to wait a further b to see an arrival, given that we have already waited a?
- $\dfrac{P(\{W > a + b\} \cap \{W > a\})}{P(\{W > a\})} = \dfrac{P(\{W > a + b\}}{P(\{W > a\})}$.
- $P(W > a + b \mid W > a) = \dfrac{e^{-\lambda(a+b)}}{e^{-\lambda a}} = e^{-\lambda b}$
- This known as the **no memory property**. The distribution of future waiting times is independent of the times already waited.

Relative Arrival Times

- Consider the case of two clumsy employees. Employee A has an average of λ accidents per year and employee B has an average of μ accidents per year.

- What is the probability that the next accident by either employee is by employee A? (Assuming their accidents are all independent of each other.)

- Let W_A be the time until employee A's next accident and let W_B be the time until employee B's next one. We then want $P(W_A < W_B)$.

Relative Arrival Times

- We calculate $P(W_A < W_B)$ by calculating the probability that, if employee A's first accident is at time *t*, employee B's first accident happens after t and we integrate this over all possible $t > 0$.

- $$P(W_A < W_B) = \int_0^\infty P(W_A < W_B \mid W_A = t) f_{W_A}(t)dt$$

- If employee A's first accident happens at time *t*, then the probability that employee B's first accident happens after this is obtained by integrating the probability density function of W_B between t and ∞.

Relative Arrival Times

- $P(W_A < W_B | W_A = t) = \int_t^{\infty} \mu e^{-\mu w_B} dw_B \ = \left[-e^{-\mu w_B}\right]_t^{\infty} = e^{-\mu t}$

- Hence. Integrating over all possible times for W_A , we obtain

$$P(W_A < W_B) = \int_0^{\infty} \lambda e^{-\lambda t} \int_t^{\infty} \mu e^{-\mu w_B} dw_B dw_A = \int_0^{\infty} \lambda e^{-\lambda t} e^{-\mu t} dt = \int_0^{\infty} \lambda e^{-(\lambda+\mu)t} dt$$

$$= \frac{\lambda}{\lambda + \mu}$$

- Intuitively, this makes sense. If A is clumsier, $\lambda > \mu$ and hence $P(W_A < W_B) = \dfrac{\lambda}{\lambda + \mu} > 0.5$

Multiple Approaches to Poisson Processes

- Consider two independent Poisson processes modelling the arrival of customers into a shop.
- The average number of customers per hour who leave without making a purchase is λ . the average number of customers who do subsequently make a purchase is μ per hour.
- What is the chance that during the next two hours, exactly three customers enter the shop and all three make a purchase?
- There are two very different approaches to answering this question.

Multiple Approaches to Poisson Processes

- Method One:

- Merge the Poisson processes to say that the total rate of customer arrivals is $\lambda + \mu$ per hour.

- The chance of exactly three arrivals during a two hour window is therefore $\dfrac{e^{-2(\lambda+\mu)}\left[2(\lambda+\mu)\right]^3}{3!}$.

- The probability that all of the next three customers to arrive make a purchase is $\left(\dfrac{\mu}{\lambda+\mu}\right)^3$.

- Combining these two probabilities, the probability of exactly three customers arriving during the two hour window and all of those three making a purchase is

$$\frac{e^{-2(\lambda+\mu)}\left[2(\lambda+\mu)\right]^3}{3!}\left(\frac{\mu}{\lambda+\mu}\right)^3 = \frac{e^{-2(\lambda+\mu)}(2\mu)^3}{3!}$$

Multiple Approaches to Poisson Processes

- Method Two:
- Keeping the Poisson processes separate, we note that the only way we get exactly three customers arriving during the two hour window and all three making a purchase is if the number of purchasing customers arriving during the two hour window is 3 and the number of non-purchasing customers arriving during the window is 0.

- The probability of three purchasing customers arriving in two hours is $\frac{e^{-2\mu}[2\mu]^3}{3!}$.

- The probability of no non-purchasing customers arriving in two hours is $\frac{e^{-2\lambda}[2\lambda]^0}{0!}$.

- The chance of both of these occurring is therefore $\frac{e^{-2\mu}[2\mu]^3}{3!}\frac{e^{-2\lambda}[2\lambda]^0}{0!}=\frac{e^{-2(\lambda+\mu)}(2\mu)^3}{3!}$.