37181 DISCRETE MATHEMATICS

Lecture 20: Trees

PLAN

- Defn: tree
- Spanning tree

DEFN: TREE

Recall that a circuit is a path $(x, v_1), \ldots, (v_n, x)$. We will also call a circuit a *cycle*.

A *tree* is an undirected graph G which is connected and has no cycles.

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A *tree* is an undirected graph G which is connected and has no cycles.

By convention, we don't allow the empty graph to be a tree.

EQUIVALENTLY

Theorem

Let G be an undirected graph. The following are equivalent (TFAE):

- *G is a tree.*
- *G has no cycles, and a simple cycle is formed if any edge is added to G.*
- *G is connected, but would become disconnected if any single edge is removed from G.*
- *G is connected and K_3 is not a* minor *of G.*
- *Any two vertices in G can be connected by a unique simple path.*

We will come back and prove just one or two of these are equivalent, leaving the rest for exercises.

PLAY AROUND WITH DEFN. HOW MANY?

$|V| = 1, 2, 3, 4, 5, \ldots.$

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Guess: what is a leaf?

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Guess: what is a leaf?

a forest?

Recall: G is a tree if it is connected and has no cycles.

Theorem

G is a tree if and only if G is connected, but would become disconnected if any single edge is removed from G.

Proof:

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Proof: Two directions.

Assume G is connected and has no cycles. Then G is connected. Suppose (for contradiction) some single edge e is removed (keeping its endpoints x, y), and G is still connected. Then there is a path p in G' from x to y, so in G there is a cycle pe. Contradiction.

Assume G is a connected graph with the property that removing a single edge always disconnects it. Then G is connected. Suppose G has a cycle. Then removing an edge on that cycle does not disconnect, contradiction. So G doesn't have any cycles. □

Theorem

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Proof: (strong induction) Let $P(n)$ be the statement that a tree with n vertices has $n-1$ edges.

True for $n = 1$ by inspection.

Assume true for all $1 \leqslant i \leqslant k$ and consider T with $k+1$ vertices. If T has no edges then it is disconnected ($k+1 \geqslant 2$), so T has an edge. Choose one (finitely many) and erase it, leaving its endpoints, to get two trees (not connected after removing an edge by previous). Call the two connected components T_1, T_2 and say T_1 has $1 \leqslant j \leqslant k$ vertices, so T_2 has $1 \leqslant k+1-j \leqslant k$ vertices. By strong induction, we know T_1 has $j-1$ edges and T_2 has $k+1-j-1 = k-j$ edges, so the original T had $(j-1)+(k-j)+1 = k$ edges (those from T_1, T_2 plus the edge we removed).

Thus by (strong) PMI the statement is true for all $n \in \mathbb{N}_+$. □

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Theorem (König's lemma)

Every infinite tree contains either

- *a vertex of infinite degree, or*
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Wikip: This proof is not generally considered to be *constructive*, because at each step it uses a proof by contradiction to establish that there exists an adjacent vertex from which infinitely many other vertices can be reached, and because of the reliance on a weak form of the *Axiom of Choice*.

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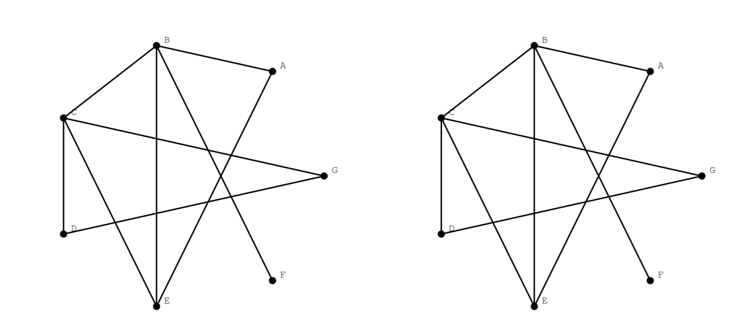

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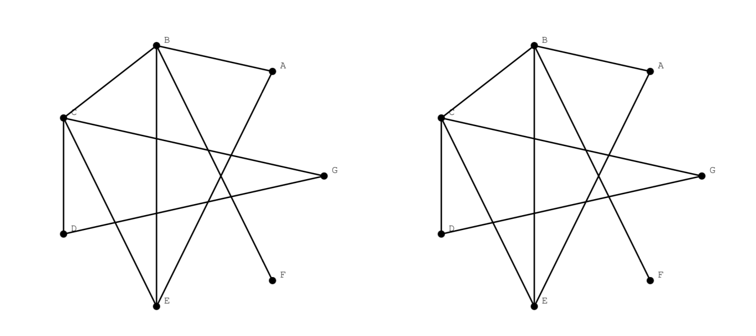

c.f. Wikip: A *spanning tree T* of an undirected graph *G* is a subgraph that is a tree which includes all of the vertices of G, with minimum possible number of edges.

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Note the statement is also true for infinite graphs, but requires more interesting logic and techniques.

In later optimisation courses you will study *efficient* algorithms to construct spanning trees. Use Big O to make precise how efficient.

NEXT TIME (LAST LECTURE!)

- Rooted trees, bracket-free expressions (pre-post-in orders)
- planar graph
- Euler's formula