37181 DISCRETE MATHEMATICS

Lecture 12: pigeonhole principle

PLAN

- pigeonhole principle

Recall:

Lemma

Let A, B be finite sets. If $f : A \to B$ is

- *1-1 then $|A| \leqslant |B|$.*
- *onto then $|B| \leqslant |A|$.*

Proof: ?

To prove the 1-1 rigorously, we need another **Axiom**

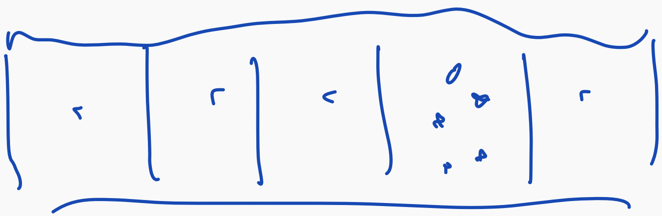
PHP

boxes

Axiom (Pigeonhole principle)

If m pigeons occupy n pigeonholes and $m > n$ then some pigeonhole has at least two pigeons in it.

m



n boxes.

Remember, this is an *axiom* like well ordering (and induction) – they seem obvious, but we can't derive them from other things.

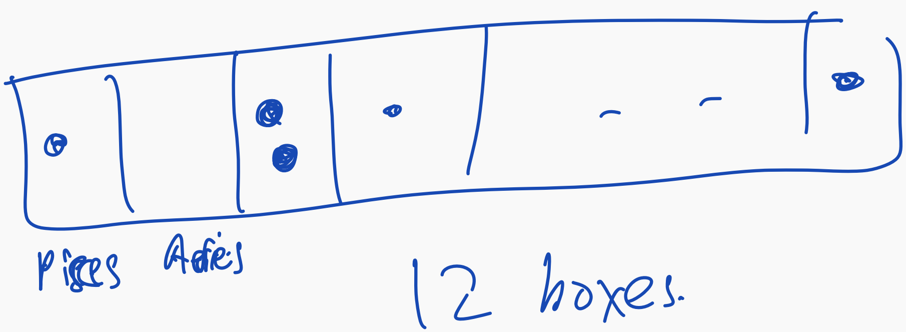
Axiom (Pigeonhole principle)

If m pigeons occupy n pigeonholes and $m > n$ then some pigeonhole has at least two pigeons in it.

Out of 13 people, what is the chance two of them have the same Western Zodiac sign?

$\boxed{100\%}$

13 people = pigeons



PHP

Axiom (Pigeonhole principle)

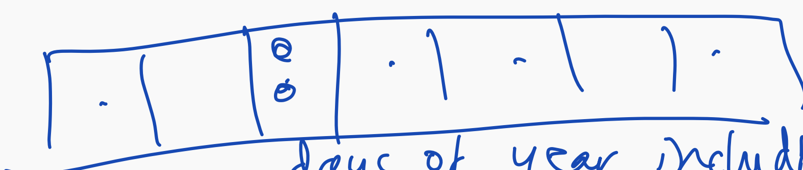
If m pigeons occupy n pigeonholes and $m > n$ then some pigeonhole has at least two pigeons in it.

Out of 13 people, what is the chance two of them have the same Western Zodiac sign?

Out of 367 people, what is the chance two of them have the same birthday?

367 people

366



days of year including Feb 29

Axiom (Pigeonhole principle)

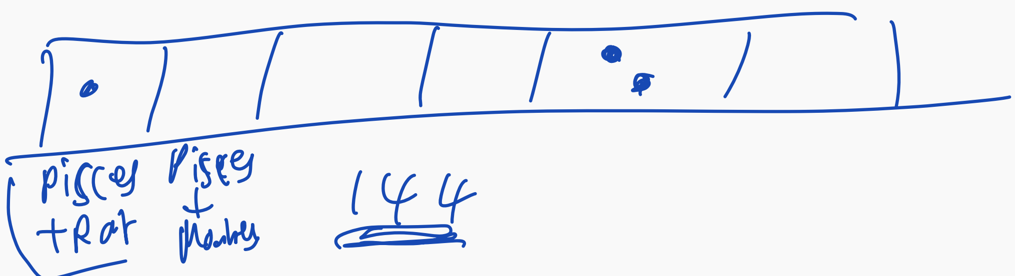
If m pigeons occupy n pigeonholes and $m > n$ then some pigeonhole has at least two pigeons in it.

Out of 145 people, what is the chance two of them have the same Western Zodiac sign AND Chinese Zodiac?

12 12

100%

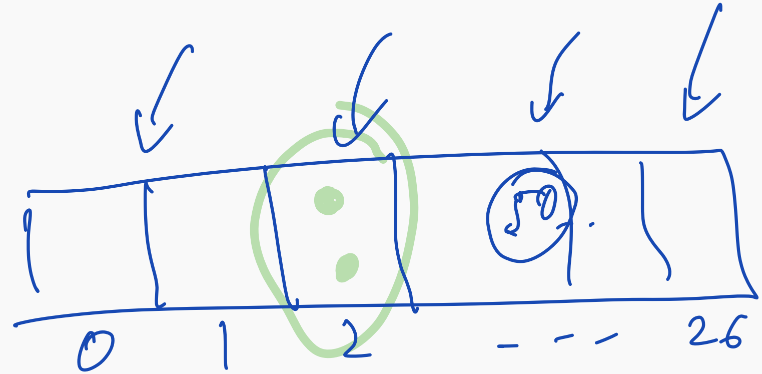
145



Let $A \subseteq \mathbb{N}_+$ with $|A| = 28$. Then A contains at least two elements with the same remainder $\mod 27$.

Proof: the pigeons are . . . number from A

the pigeonholes (boxes) are . . .



remainder on div by 27

and the rule for placing the pigeons into the boxes is:

compute remainder on div by 27

$0 \leq r < 26.$

Set $A \subseteq \{1, \ldots, 100\}$ and $|A| = 11$.

distinct

If 11 integers are chosen from $\{1, 2, 3, \ldots 100\}$ then at least two, say x and y, are such that

$x \neq y$

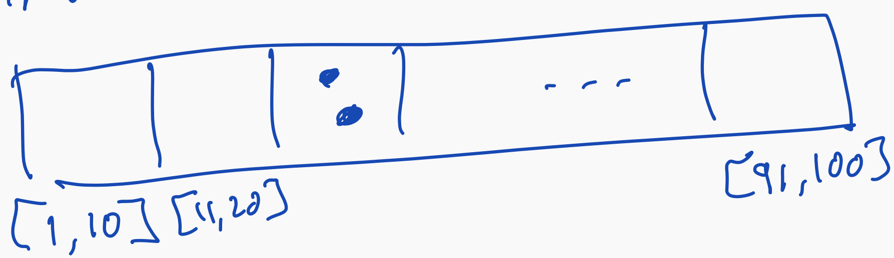
$$|x - y| \leqslant 9$$

Proof: the pigeons are ... 11 numbers

10 intervals

the pigeonholes are ...

intervals $[10i+1, 10i+10]$



and the rule for placing the pigeons into the boxes is:

for $a \in A$, find $0 \leq i \leq 9$ so that $a \in [10i+1, 10i+10]$

EX

If 11 integers are chosen from $\{1, 2, 3, \ldots 100\}$ then at least two, say x and y, are such that

$$|x - y| \leqslant 9$$

Proof: the pigeons are . . .

the pigeonholes are . . .

and the rule for placing the pigeons into the boxes is:

EX

$A \subseteq \{1, \dots, 100\}$, $|A| = 11$:

m

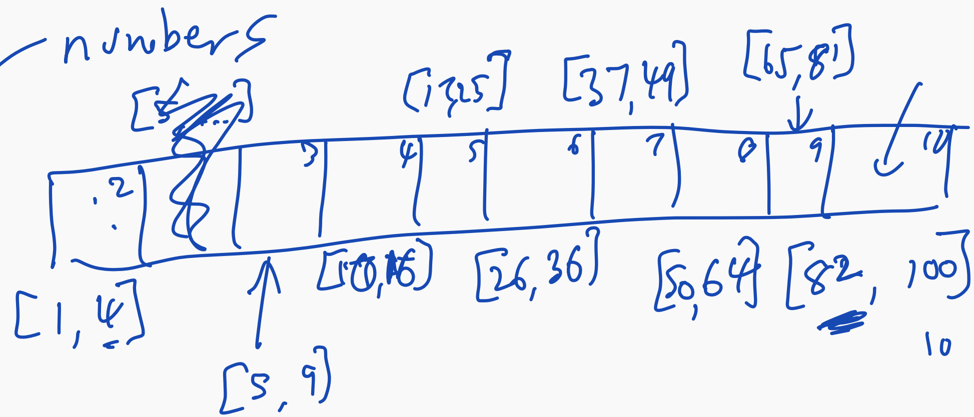
If 11 integers are chosen from $\{1, 2, 3, \dots, 100\}$ then at least two, say x and y, are such that

distinct $x \neq y$,

$$|\sqrt{x} - \sqrt{y}| < 1$$

Proof: the pigeons are . . . numbers

11 numbers



the pigeonholes are . . .

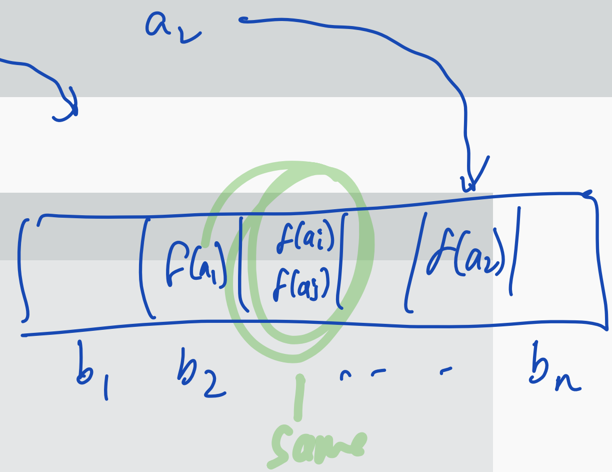
9 boxes

and the rule for placing the pigeons into the boxes is:

for $a \in A$ take $\sqrt{a}$

then find interval so that

$\sqrt{a} \in [\dots]$



Recall:

Lemma

Let A, B be finite sets. If $f : A \to B$ is

- *1-1 then $|A| \leqslant |B|$.*
- *onto then $|B| \leqslant |A|$.*

Proof: 1-1: Suppose $|A| > |B|$.

let A be pigeons, B boxes and rule place the element $a \in A$ in the box $f(a) \in B$

By PHP, $\exists\, a_i, a_j \in A$, $a_i \neq a_j$, so that $f(a_i) = f(a_j)$

$\therefore$ f is not 1–1.

So by Contrapositive:

f 1–1 $\rightarrow$ $|A| \leq |B|$.

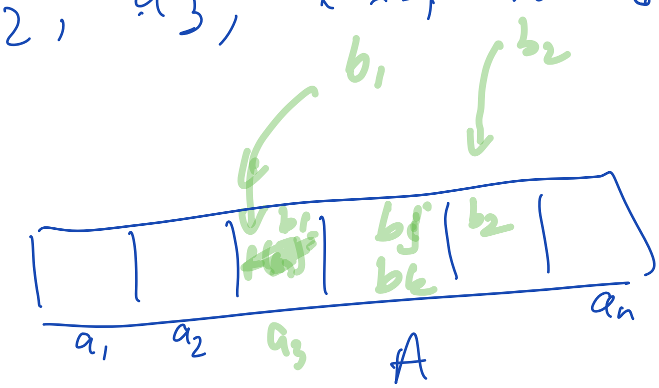
~~Let f onto function.~~

Onto: Suppose for contradiction $|B| > |A|$, ~~and f onto.~~

Since A is finite, we can write its elements in order

$a_1, a_2, a_3, \ldots, a_n$.

Boxes: A



Pigeons: B

Rule $b \in B$ goes into a_i box if $f(a_i) = b$ and i is smallest such subscript.

Since f is onto, for each $b \in B$ there is always one a_i.

labelled a_i

By PHP, some box labelled a_i will contain at least <u>two</u> elements of B:

Say $\underline{b_j}$, $\underline{b_k}$, $b_j \neq b_k$.

But this means

$$f(\underline{a_i}) = b_j$$
$$f(\underline{a_i}) = b_k$$

So f is not a well defined function

$\therefore$ Contradiction. $\square$

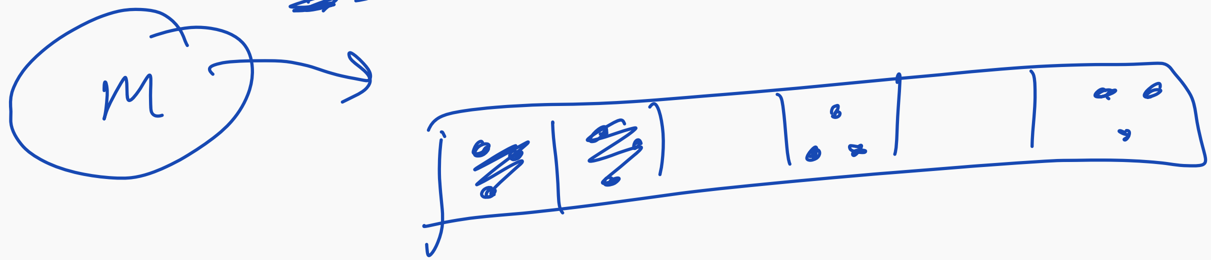
✓✓

Generalised

Axiom (Pigeonhole principle)

Let $m, n, k \in \mathbb{N}_+$.

If m pigeons occupy n pigeonholes and $m > kn$ then some pigeonhole has more than k pigeons in it.



n

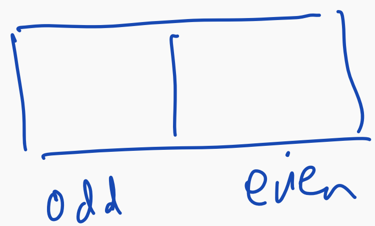
kn.

Ex: Use PHP to show that if $S \subseteq \mathbb{N}_+$ and

1. $|S| \geqslant 3$ then S contains two distinct elements x, y such that $x + y$ is even.

3 numbers.
= pigeons

2 boxes:



By PHP: some box has 2 numbers in it

If 2 numbers x, y in the "ODD box"

then $x+y$ is even.

If 2 numbers in "EVEN" box
x, y.
$x + y$ even.

Ex: Use PHP to show that if $S \subseteq \mathbb{N}_+$ and

2. $|S| > 6$ then S contains three distinct elements x, y, z such that $x + y + z$ is a multiple of 3.

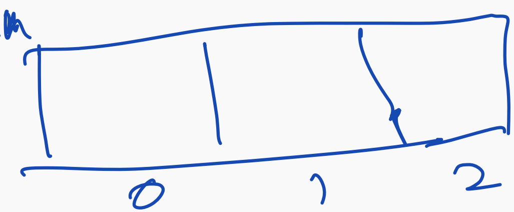
$k = 2$

$n = 3$

boxes: remainder on division by 3.

pigeons: S. $|S| = m > 6 = k \cdot n$

rule: $s \in S$ compute its remainder on division by 3



By GPHP, some box has at least 3 distinct numbers from S

x, y, z

If "0" box: $x+y+z$ is divisible by 3.

If "1" box: $x = 3p+1$
$y = 3q+1$
$z = 3r+1$

so $x+y+z = 3(p+q+r)+3$ ✓✓

If in "2" box: $x = 3p+2$
$y = 3q+2$
$z = 3r+2$

$x+y+z =$
$3(p+q+r)+6 = 3(p+q+r+2)$ ✓✓

NEXT

- how to count
- inclusion-exclusion
- permutations and combinations