37181 DISCRETE MATHEMATICS

Lecture 9: relations, functions

PLAN

- relations
- functions
- one-to-one
- onto

If A, B are sets we can define a new symbol (a, b) where $a \in A$ and $b \in B$.

This symbol is not the same as $\{a, b\}$, it is a new symbol. Also it is not the same as (b, a), the symbol has an *order*.

We call it an *ordered pair*.

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Define $A \times B = \{(a, b) \mid a \in A, b \in B\}$. Cartesian Product

Eg: If $A = \{1, 2, 3\}$ and $B = \{d, e\}$ then $A \times B =$ $\{(1,d), (1,e), (2,d), (2,e), (3,d), (3,e)\}$

AXIOM: If A, B are sets then so is $A \times B$ — actually can prove it from other Axioms in ZF.

ORDERED PAIRS, RELATIONS

A subset of $A \times B$ is called a *relation* from A to B.

We often use the notation $\mathscr{R}$ to denote a relation.

Eg: Let $A = \{1, 2, 3, 4\}$ and define $\mathscr{R} \subseteq A \times A$ by
$\mathscr{R} = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$.

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We write $a\mathscr{R}b$ if $(a, b) \in \mathscr{R}$, and say "a is related to b". So for example $1\mathscr{R}3$.

What is another notation you could use for this relation?

Let $A = \{1, 2, 3, 4\}$ define a relation $\mathscr{R} \subseteq A \times A$ which means "$\geqslant$"

$$R = \{ (1,1), (2,1), (2,2), (3,1), (3,2), (3,3), (4,1), (4,2), (4,3), (4,4) \}$$

WE HAVE SEEN THIS BEFORE

$a \equiv b \mod d$

$d \mid b-a$

Recall Homework Sheet 2 you learned the definition $\equiv \mod d$.

Let $\mathbb{Z}$ be our set and define $\mathscr{R}_d \subseteq \mathbb{Z} \times \mathbb{Z}$ by $a \mathscr{R}_d b$ if $a \equiv b \mod d$.

Recall Homework Sheet 2 you learned the definition $\equiv \mod d$.

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Ex: Write down some elements $a \in \mathbb{Z}$ such that $a \mathscr{R}_5 1$:

$\equiv 1 \mod 5$

$R_5 = \{(6,1), (11,1), (16,1), (26,1), (1,1), (-4,1), (-9,1) \ldots$

Definition

Let A be a set. Then $\mathscr{R} \subseteq A \times A$ is

- *reflexive* if for all $a \in A$, $a\mathscr{R}a$
- *symmetric* if for all $a, b \in A$, $a\mathscr{R}b$ implies $b\mathscr{R}a$
- *antisymmetric* if for all $a, b \in A$, $a\mathscr{R}b$ and $b\mathscr{R}a$ implies $a = b$
- *transitive* if for all $a, b, c \in A$, $a\mathscr{R}b$ and $b\mathscr{R}c$ implies $a\mathscr{R}c$

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Ex: Let $A = \{1, 2, 3\}$ and

$$\mathscr{R} = \{(1,1),(2,2),(3,1),(1,3),(2,3),(3,2)\}.$$

Decide which of the four properties (reflexive, symmetric, antisymmetric, transitive) $\mathscr{R}$ satisfies.

↓

$(1,3) \; , \; (3,2) \in R$

$\Rightarrow (1,2) \in R$?

Definition

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- *symmetric* if for all $a, b \in A$, $a\mathscr{R}b$ implies $b\mathscr{R}a$
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Ex: Construct an example (that means tell me a set A and some subset of $A \times A$) of a relation which is

- *both* symmetric and antisymmetric
- *neither* symmetric nor antisymmetric

transitive? (3,1) (1,2)

$\not\Rightarrow$ (3,2)

DEFINITIONS

These notions are extremely useful throughout mathematics.

For now, you should feel good if you can read the very abstract definitions (written in logic and set theory notation) and write down examples, prove/disprove some relation has them.

This will show you are “getting it” in this course.

DEFINITIONS

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(Homework sheet 2 Question 5: you already showed $\equiv \mod d$ is reflexive, symmetric and transitive and not symmetric.)

anti

Ex

$a \equiv a \mod d$

$a - a = 0$

$d \mid 0$

DEFINITIONS

Definition

Let A be a set. Then $\mathscr{R} \subseteq A \times A$ is

- an *equivalence relation* if it is reflexive, symmetric and transitive
- a *partial order* if it is reflexive, antisymmetric and transitive

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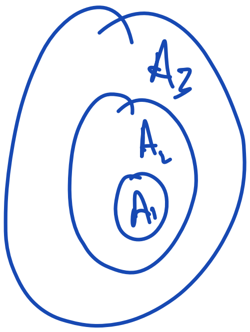
- a *partial order* if it is reflexive, antisymmetric and transitive

Eg: "$\equiv \mod d$" is an equivalence relation on $\mathbb{Z}$.

Ex: Show that "$\leqslant$" is a partial order on $\mathbb{Z}$.

Ex: Show that if A is a set, then "$\subseteq$" is a partial order on $\mathscr{P}(A)$.



$$A_1 \subseteq A_2 \subseteq A_3$$

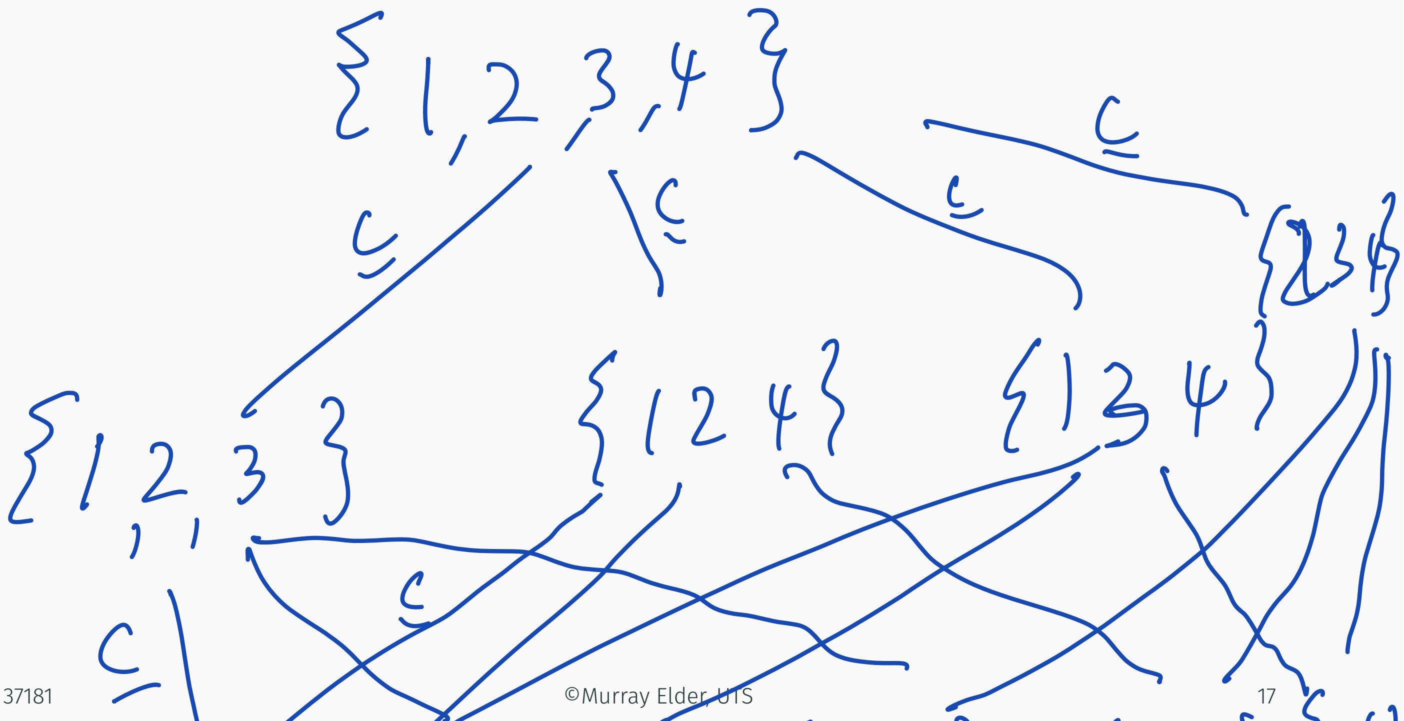
$$\rightarrow A_1 \subseteq A_3.$$

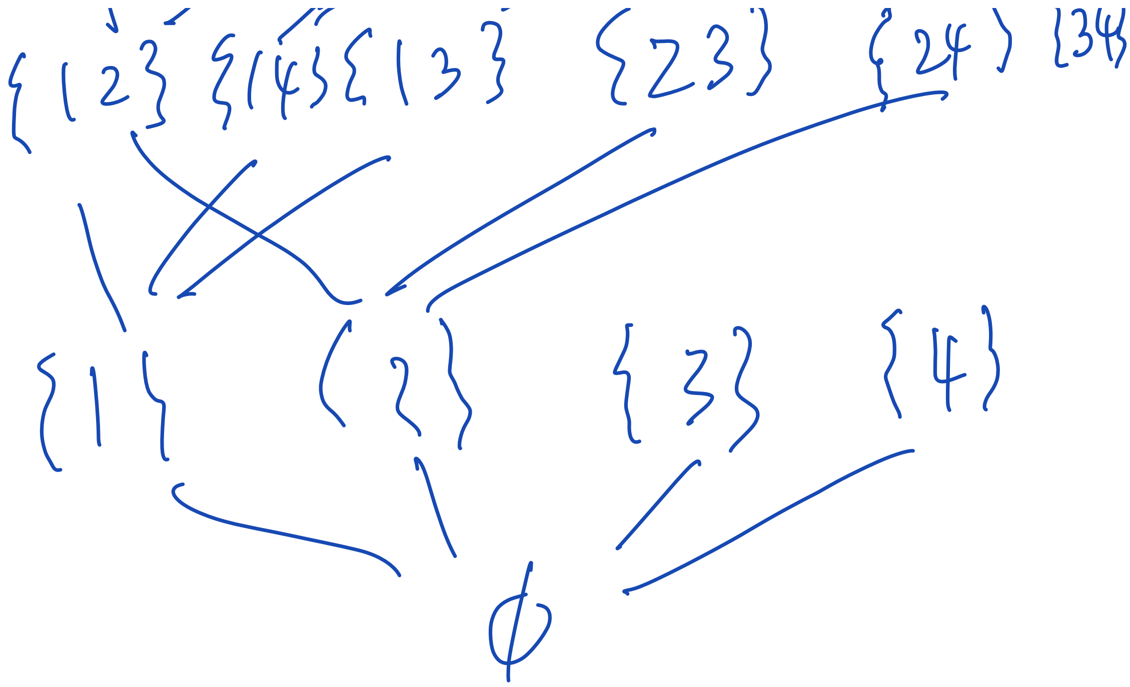
$$A_1 \; A_2 \; A_3 \; \ldots \in \wp(A)$$

$$A_i \subseteq A_i \quad \forall i \rightarrow \text{reflexive.}$$

Given a partial order on a set we can draw a nice picture called a *Hasse diagram*. Here is an example:

$A = \{1, 2, 3, 4\}$, relation is "$\subseteq$" on $\mathscr{P}(A)$.





{12} {14} {13} {23} {24} {34}
{1} {2} {3} {4}
ϕ

A *function* from A to B is a relation $f \subseteq A \times B$ in which every element of A appears exactly once as the first component of an ordered pair in the relation.

$$\forall a \in A \quad \exists b \in B \left[(a,b) \in f \right]$$

$$\wedge \left[\forall a \in A\ \forall b_1, b_2 \in B \quad (a, b_1) \in f,\ (a, b_2) \in f \Rightarrow b_1 = b_2 \right]$$

A *function* from A to B is a relation $f \subseteq A \times B$ in which every element of A appears exactly once as the first component of an ordered pair in the relation.

Since for each $a \in A$ we have exactly one $(a, b) \in f$ we can also use the notation $f(a) = b$, and we write $f : A \to B$.

$(a, b) \in f$

is denoted

$f(a) = b$

$f = R$

x

$R(a) = ?$

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Eg: Let $S =$ the set of all students at UTS and $f \subseteq S \times \mathbb{N}$ where (s, n) means n is a student ID number for student s.

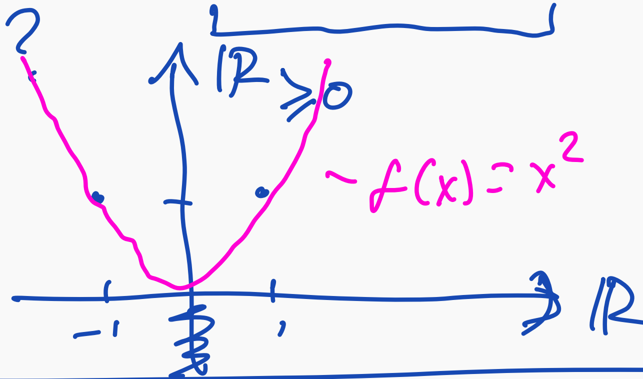
What if f was not a function? — Some $s \in S$ have maybe NO number or ≥ 2 number.

What if $(s, 13645)$ and $(t, 13645)$ were both in f? ← this is ok for a function.

$f \subseteq \mathbb{R} \times \mathbb{R}_{\geq 0}$

Eg: sets are $A = \mathbb{R}$, $B = \mathbb{R}_+ \cup \{0\}$, relation is $\{(x, x^2) \mid x \in \mathbb{R}\}$.

Is this a function? ✓✓



Also $f \subseteq \mathbb{R} \times \mathbb{R}$ $(= \mathbb{R}^2)$

$f = \{(x, x^2) \mid x \in \mathbb{R}\}$ is another function (different target set).

Eg: Define $f : \mathbb{R} \to \mathbb{Z}$ by

floor function

$$f(x) = \lfloor x \rfloor = \text{ the biggest integer less than or equal to } x.$$

Similarly we have $g : \mathbb{R} \to \mathbb{N}$ by

ceiling function

$$g(x) = \lceil x \rceil = \text{ the least integer greater than or equal to } x.$$

$$\lfloor 7.2 \rfloor = 7$$

~~6~~

$$\lceil 7.2 \rceil = 8$$

FUNCTIONS

Eg: Define $f : \mathbb{R} \to \mathbb{Z}$ by

$$f(x) = \lfloor x \rfloor = \text{ the biggest integer less than or equal to } x.$$

Similarly we have $g : \mathbb{R} \to \mathbb{N}$ by

$$g(x) = \lceil x \rceil = \text{ the least integer greater than or equal to } x.$$

Eg: Let $h : \mathbb{N} \to \mathbb{N}$ defined by

$$h(n) = \left\lceil \frac{n}{2} \right\rceil + 7.$$

If $n =$ your age, compute $h(n)$.

https://en.wikipedia.org/wiki/Age_disparity_in_sexual_relationships. Should it be $\lceil \cdot \rceil$ or $\lfloor \cdot \rfloor$?

Definition

Let $f : A \to B$ be a function from a set A to a set B. We say f is *one-to-one* (or 1-1) if

$$\forall x \in A \forall y \in A[f(x) = f(y) \to x = y].$$

We want the student number function to be one-to-one.

Show that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 5x + 3$ is one-to-one.

Suppose $f(x) = f(y)$ ($x, y \in \mathbb{R}$ some)

then $5x + 3 = 5y + 3$

subtract 3 b.s. $5x = 5y$

divide b.s. by 5 $x = y$

DIRECT

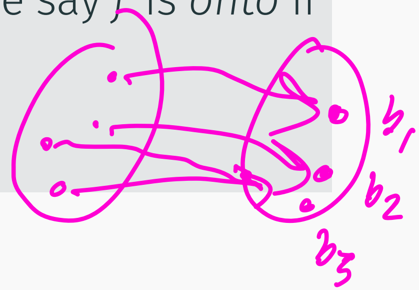
$x \neq y \Rightarrow f(x) \neq f(y)$

contrapositive

Definition

Let $f : A \to B$ be a function from a set A to a set B. We say f is *onto* if

$$\forall b \in B \exists a \in A[f(a) = b].$$



Ex: Show that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 5x + 3$ is onto

Given $b \in \mathbb{R}$

$\exists \frac{b-3}{5} \in \mathbb{R}$ $\quad f\left(\frac{b-3}{5}\right) = 5\left(\frac{b-3}{5}\right)+3$

$= b-3+3$

$= b$

Definition

Let $f : A \to B$ be a function from a set A to a set B. We say f is *onto* if

$$\forall b \in B \exists a \in A[f(a) = b].$$

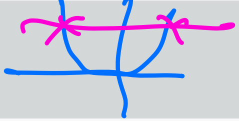
(surjective)

Ex: Show that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 5x + 3$ is onto

Ex: Show that $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$ is onto and not 1-1.

Onto: $\forall b \in \mathbb{Z}$ $\exists b \in \mathbb{R}$ $\lceil b \rceil = b$

$\lceil 2.5 \rceil = 3$ $\lceil 2.7 \rceil$ and $2.5 \neq 2.7$ not one to one.



Setting up our definitions using logical statements like this, it is easy to prove examples satisfy them or not.

$$\neg\forall x\forall y \in A[f(x) = f(y) \to x = y]$$
$$\leftrightarrow \exists x\exists y \in A[f(x) = f(y) \wedge x \neq y]$$

$$\neg\forall b \in B\exists a \in A[f(a) = b]$$
$$\leftrightarrow \exists b \in B\forall a \in A[f(a) \neq b]$$

Eg: $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$

Let $A = \{a, b, c, d, e\}, B = \{b, d, e\}, C = \{f, g, a\}$. Give examples of functions

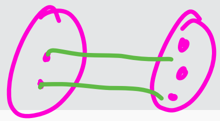
1. $f : A \to B$ which is onto and not 1-1
2. $g : A \to B$ which is 1-1 and not onto
3. $h : A \to B$ which is both 1-1 and onto
4. $i : B \to C$ which is onto and not 1-1
5. $j : B \to C$ which is 1-1 and not onto
6. $k : B \to C$ which is both 1-1 and onto

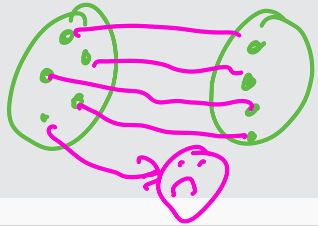
|A| denotes the number of elements in A.

Lemma

Let A, B be finite sets. If $f : A \to B$ is a function and is

1. • 1-1 then $|A| \leqslant |B|$.
   • onto then $|B| \leqslant |A|$.





Proof.

1. Contrapositive
   suppose $|A| > |B|$
   then $\{f(a_1), f(a_2), \ldots, f(a_{|A|})\} \subseteq B$

   so there must be repetition in this subset.

   $\therefore \exists\ 1 \leq i < j \leq |A|$

   so that

2.

□

[illegible]

NEXT

$a_i \neq a_j$ $f(a_i) = f(a_j)$

Exercize.

Next lecture:

- Ackermann's function
- bijection
- countable/uncountable

a function which is both one-to-one and onto.

In particular, if A, B finite sets, $f: A \to B$ is bijection then $|A| = |B|$.