Functional Analysis

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Part I

Fundamentals

Chapter 1

Topological linear spaces

Functional analysis was originally developed to study *functionals*; maps whose domain consisted of functions and whose codomain was $\mathbb{R}$ or $\mathbb{C}$. Eventually it was discovered that certain classes of linear spaces proved extremely effective at modelling function spaces (and therefore provided powerful tools to study functionals). Because of this, functional analysis eventually generalized to the study of *complete topological linear spaces* and *infinite dimensional linear spaces*; function spaces often tend to be linear spaces in both of these categories, so the specific study of functionals flows natrually from functional analysis.

Functional analysis regarding topological linear spaces is extremely compatible with real, complex, and vector analysis since the spaces of these disciplines ($\mathbb{R}^n$ and $\mathbb{C}^n$) are in fact finite dimensional topological linear spaces.

We will first look at *topological linear spaces*, indeed, a topology on the linear space will be necessary in order to interpret series of vectors; without some topology there is no way to interpret convergence!

Definition 1.1 (Topological linear space (TLS))**.** A *topological linear space (TLS)* is a linear space such that F is a topological field, and there is a topology considered on V such that $+, \cdot$ are continuous (taken as product topologies)

We could indeed go down the path of studying the properties of TLSs with a generic topological space, however function spaces typically have a norm,

Normed linear spaces form a TLS where the underlying topological space is a metric space; this makes normed linear spaces very desirable.

Proposition 1.1. Normed linear spaces induce a metric space.

Furthermore, the properties of linear spaces give the underlying metric space another nice property.

Proposition 1.2. Normed linear spaces induce a translation and scaling invariant distance function. Normed linear spaces are separable iff there is a countable subset whose span is dense in the space

1.1 Banach spaces

In mathematical analysis, $\mathbb{R}^n$ is particularly rich in mathematical analysis since it forms a complete space. For the sake of generalizing the properties of $\mathbb{R}^n$ and because completeness will ensure that we cannot use a vector series to converge objects that exist outside of the space. When studying infinite dimensional linear spaces, we rely on completeness so that the notion of the *Schauder basis* works properly.

We have seen that normed linear spaces automatically introduce extremely nice topological properties (i.e forms a metric space), combining the notion of a norm and completeness results in a *Banach space*; a complete normed linear space. Many function spaces are simply infinite dimensional Banach spaces, and indeed the familiar $\mathbb{R}^n$ is a Banach space.

Definition 1.2 (Banach space)**.** A *Banach space* is a normed linear space $(V, \|\cdot\|)$ such that its induced metric space (V, d) (where $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$) is a complete metric space.

Interesting, one can check that a normed linear space is Banach when absolute convergence implies convergence.

Proposition 1.3. Let X be a normed linear space, X is a Banach space iff any absolute convergent series is a convergent series.

1.2 Hilbert space

Hilbert space; complete inner product spaces.

Definition 1.3 (Hilbert space)**.** A *Hilbert space* is an inner product space $(V, \langle\cdot,\cdot\rangle)$ such that its induced metric space (V, d) (where $d(\mathbf{u}, \mathbf{v}) = \sqrt{\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v}\rangle}$) is a complete metric space.

Hilbert spaces are notorious as one of, if not, the closest class of abstract spaces to Euclidean space.

Chapter 2

Infinite dimensional linear spaces

We will explore "infinite" analogues of various fundamental linear algebra concepts that can exist in topological linear spaces.

2.1 Infinite linear combinations

Studying linear combinations is fundamental for analyzing linear spaces since it is a precursor to the notion of a *basis*, which is perhaps the ultimate tool for studying a linear space.

Definition 2.1 (Infinite linear combination)**.** In a TLS and for a set of vectors and set of coefficients, an *infinite linear combination* is a vector series of the following form.

$$\sum_{k=1}^{\infty} c_k \mathbf{b}_k$$

2.2 Closed linear span

Given the fact that we are working in a topological linear space, we can and examine all the limit points of this set (i.e the infinite series that could potentially be made.

Definition 2.2 (Closed linear span)**.** Let V be a TLS, a *closed linear span of* S is the closure in $(V, \mathcal{T})$ of the span span(S)

$$\mathrm{cl}(\mathrm{span}(S))$$

Due to the continuity of $+, \cdot$, this set is indeed its own linear subspace.

Proposition 2.1. Closed linear spans a linear subspace

2.3 Schauder basis (functional analysis)

We motivate our discussion by the following proposition that holds for a Hamel basis in a finite dimensional space; the whole idea of a Schauder basis is to find such a representation in our infinite dimensional spaces.

Proposition 2.2. Let V be an n-dimensional linear space with basis $(\mathbf{b}_k)_{k=1}^n$, then for every vector $\mathbf{v}$ there exists a unique sequence $(c_k)_{k=1}^n$ such that the following holds

$$\sum_{k=1}^{n} c_k \mathbf{b}_k$$

Because this is a finite linear combination, the order of terms may be permuted without effect since vector addition is commutative. However an infinite linear combination however may not obey the property of unconditional convergence; it is possible that permuting terms of the series can change its limit or even make it diverge! Because of this, we require a countable set of vectors obeying a similar property to be either linearly ordered or a sequence.

Definition 2.3 (Schauder basis of a linear space)**.** Let V be a TLS, a *Schauder basis of* V is a sequence of distinct vectors $(\mathbf{b}_n)$ such that for any vector $\mathbf{v} \in V$ there exists a unique $(c_n)_{n=1}^\infty$ such that the following holds.

$$\sqsubseteq = \sum_{n=1}^{\infty} c_n \mathbf{b}_n$$

2.4 ℓ^p spaces

Our knowledge of TLSs and infinite dimensional linear spaces is sufficient to begin developing some concrete function spaces. One of the most basic examples of infinite dimensional linear spaces are sequence spaces, among which the ℓ^p are a special subspace that will be seen to form.

Definition 2.4 (Sequence space)**.** Given a field F, the space $F^{\mathbb{N}}$ represents all sequences $\{x_n\}_{n\in\mathbb{N}} \in F$ with vector addition and multiplication defined as such. $\{x_n\}_{n\in\mathbb{N}} + \{y_n\}_{n\in\mathbb{N}} = \{x_n + y_n\}_{n\in\mathbb{N}}$ $a\{x_n\}_{n\in\mathbb{N}} = \{ax_n\}_{n\in\mathbb{N}}$

One can naturally imagine a product topology for this linear space, and one can prove that this is frechet, complete, and metrizable. Unfortunately however, there is no continuous norms that can be made on this space, hence unfotunately it cannot be a normed linear space.

If we work backwards from an ideal norm, reminiscent of the EUclidean norm we are familiar with, perhaps we can find a normed subspace of this space.

2.4.1 p-norm

In linear algebra one can study spaces $\mathbb{R}^n$ (or F^n) using a finite p-norm. When dealing with sequence spaces, we can equip them with similarly with a *p-norm*.

Definition 2.5 (p-norm of $F^{\mathbb{N}}$)**.** When $p \geq 1$

$$\|\{x_n\}\|_\infty = (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}}$$

$$\|\mathbf{x}\|_\infty = \max_{n\in\mathbb{N}}(|x_n|)$$

It is customary to check that this is actually a norm for $p \geq 1$.

For now, we assume that we are considering sequences such that its p-norm in the natural product topology converges to begin with. However notice that the triangle inequality fails to hold for $p < 1$, and hence cannot be a norm for $p < 1$. But as it turns out, the p-norm is nice enough to act as a norm for $p \geq 1$

2.4.2 ℓ^p space

The main issue with the p-norm is that of convergence; on a general sequence space, the p-norm may very well diverge! Therefore we'll consider a linear subspace for which the p-norm does converge; this is the notion of an ℓ^p space.

Considering that we want a linear space on sequences that agrees with some p-norm as its norm. Our p-norm satisfies the definition of a norm with $p \geq 1$, however the p-norm doesn't converge for every sequence. Therefore we admit into our subspace the sequences for which the chosen p-norm converges, which translates to sequences such that that inner sum $\sum_{k=1}^{\infty} |[\mathbf{x}]_k|^p$ converges.

Definition 2.6 (ℓ^p space)**.** An ℓ^p *space of* F is a linear subspace of $F^{\mathbb{N}}$ that is the normed linear space $(F, +, \cdot, \ell^p(F), \|\cdot\|_p)$ with the following.

$$\ell^p(F) = \{\{x_n\} \in F^{\mathbb{N}} : \|\{x_n\}\| < \infty\}$$

Since we have constructed this space with the intention of creating a normed linear space on sequences ona complete field, we immediately know the following, using our construction as a proof.

Proposition 2.3. $p \geq 1$, then ℓ^p is a Banach space.

What's more is the following.

Proposition 2.4. ℓ^2 is a Hilbert space.

The $\langle\{x_n\}, \{y_n\}\rangle = \sum_{n=1}^{\infty} |x_n y_n|$ is a valid inner product in ℓ^2 since the fact that if follows the Cauchy-Schwarz inequality can be used to not only guarantees its convergence, but also guarantees it follows the triangle inequality, hence proving it is an inner product.

2.5 L^p spaces

Now that we have familiarity with some infinite dimensional linear spaces, we can begin studying functionals.

In real analysis we consider $\mathbb{R}$ and the functions upon it; functional analysis will consider function spaces and the *functionals* upon it.

Definition 2.7 (Essential supremum).

$$\text{esssup} f = \inf\{c \in Y : \mu(\{x \in X : f(x) \leq c\}) = 0\}$$

Definition 2.8 (L^p-norm).

$$\|f\|_p = (\int_X |f|^p d\mu)^{\frac{1}{p}}$$

$$\|f\|_\infty = \text{esssup}|f|$$

Definition 2.9 (L^p space).

$$L^p(X, \Sigma, \mu) = \{f : X \to \mathbb{R} : \|f\|_p < \infty\}$$

To develop some inequalites on L^p spaces, we first prove a simple lemma that requires nothing than basic real analysis and optimization.

Lemma 2.1 (Young's inequality).

$$\frac{1}{p} + \frac{1}{p} = 1$$

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Theorem 2.1 (Hölder's inequality).

$$\frac{1}{p} + \frac{1}{p} = 1$$

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

The triangle inequality is one of the most powerful tools in mathematical analysis, and indeed it exists on L^p spaces.

Theorem 2.2 (Minkowski's inequality).

$$p \in [1, \infty) \cap \{\infty\}$$

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

2.5.1 Othogonal Schauder basis of an L^p spaces

Definition 2.10 (Chebyshev polynomials)**.** The *Chebyshev polynomials (first kind)* are a sequence of polynomials $(T_n)_{n\in\mathbb{N}}$ relating to cosine relations of an angle with factor n.

$$T_n(\cos(\theta)) = \cos(n\theta)$$

- $T_0 = 1$
- $T_1 = x$
- $T_{n+1} = 2xT_n - T_{n-1}$

The *Chebyshev polynomials (second kind)* are a sequence of polynomials $(U_n)_{n\in\mathbb{N}}$ relating to sine relations of an angle with factor n.

$$U_{n-1}(\cos(\theta)) = \frac{\sin(n\theta)}{\sin(\theta)}$$

- $U_0 = 1$
- $U_1 = 2x$
- $U_{n+1} = 2xU_n - U_{n-1}$

Definition 2.11 (Legendre polynomials)**.** The *Legendre polynomials* are a sequence of polynomials $(P_n)_{n\in\mathbb{N}}$ forming an orthogonal basis for $L^2[-1,1]$ with weight 1.

Definition 2.12 (Hermite polynomial (Physicist's))**.**

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})$$

The Hermite polynomials are an orthogonal Schauder basis for $L^2(\mathbb{R})$

In fact the trigonometric functions form an orthogonal basis for $L^2(I)$, a result called Carleson's theorem; in this way functional analysis gives the justification for the methods of Fourier analysis!

2.5.2 Riesz-Fischer theorems

The Riesz-Fischer theorem gives the central motivation for studying L^p spaces rather than spaces of Riemann integrable functions, because it gives assurance that the L^p spaces are complete.

Theorem 2.3 (Riesz-Fischer theorem I)**.**

$$p \in [1, \infty) \implies L^p \text{ is a Banach space}$$

Theorem 2.4 (Riesz-Fischer theorem II)**.**

$$L^2 \text{ is a Hilbert space}$$

These results are why L^p spaces are often preferred over spaces of Riemann integrable functions; because they are complete spaces.

Theorem 2.5 (Weierstrass approximation theorem (L^p space))**.** *Let I be a compact interval of $(\mathbb{R}, \mathcal{E})$, the set of polynomials on I is dense in $L^2(I)$ and $L^1(I)$*

2.6 C^k spaces

C^0 space (continuous functions) C^k space (k times continuously differentiable functions) C^∞ space (infinitely differentiable functions) C^ω space (analytic functions)

The Weierstrass approximation theorem is often covered in a course on real analysis; and now we will return to it by restating it equivalently by means of

Theorem 2.6 (Weierstrass approximation theorem)**.** *Let I be a compact interval of $(\mathbb{R}, \mathcal{E})$, the set of polynomials on I is dense in $C(I)$*

Chapter 3

Operator theory

Definition 3.1 (Operator)**.** An *operator* is map between two TLSs with the same field $T : X \to Y$.

Example 3.1 (Fourier transform)**.**

$$\mathcal{F} : L^1(\mathbb{R}) \to L^\infty(\mathbb{R})$$

A type of frequently occuring operator is a bounded and continuous operator on normed linear spaces; indeed the Fourier transform This gives rise to an equivalent definition for a bounded linear operator when considering operators between normed linear spaces.

Definition 3.2 (Bounded linear operator (normed linear space))**.** A *bounded linear operator* is a linear operator $T : X \to Y$ such that there exists some c such that for all $\mathbf{x} \in X$ we have $\|T\mathbf{x}\|_Y \leq c\|\mathbf{x}\|_X$ In other words, it is Lipschitz continuous.

Futhermore, one can equivalently prove that BLOs are precisely the linear operators that map bounded sets to bounded sets; under this condition we can generalize BLOs to any TLS.

Definition 3.3 (Bounded linear operator (BLO))**.** A *bounded linear operator (BLO)* is a linear operator $T : X \to Y$ such that $T(U)$ is bounded in $(Y, \mathcal{T}_Y)$ when U is bounded in $(X, \mathcal{T}_X)$

Proposition 3.1. Let $T : X \to Y$ be an linear operator on normed linear spaces, then T is a BLO iff it is continuous.

Corollary 3.1. Let T be a bijective BLO, then T^{-1} is linear. Let $T : X \to Y$ be a bijective operator, then T^{-1} is continuous if $\|T\mathbf{x}\|_Y \leq c\|\mathbf{x}\|_X$.

Corollary 3.2. Let $T : X \to Y$ be a BLO between normed linear spacs, then T preserves linear spaces T preserves convex sets

Definition 3.4 (Functional)**.** A *functional* is a BLO between normed linear space and its field $T : X \to F$.

Definition 3.5 (Projection)**.** A *projection* is an idempotent operator. An operation is idempotent if subsequent compositions of the operation do not change its result.

$$P^2 = P$$

Projections only have ± 1 as eigenvalues

3.1 Operator norms

The "Lipschitz condition" that all BLOs between normed linear spaces obey implies that the quantity $\frac{\|T\mathbf{x}\|_Y}{\|\mathbf{x}\|_X}$ is bounded above by some Lipschitz constant; the Dedekind completeness of $\mathbb{R}$ means that $\{\frac{\|T\mathbf{x}\|_Y}{\|\mathbf{x}\|_X} \in \mathbb{R} : x \in X\}$ has a supremum; this value can be used as the smallest possible Lipschitz constant, and we call this constant the *operator norm*.

Definition 3.6 (Operator norm)**.** Let T be an operator, then its *operator norm* is defined by the following expression

$$\|T\| = \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\|T\mathbf{x}\|_Y}{\|\mathbf{x}\|_X}$$

As it turns out (and I not-so-suavely alluded to by calling it the operator NORM) operator norms are indeed a norm! We denote these operator spaces by B(X,Y)

When the codomain of the operator is a complete space, the operator space is also complete.

Compositions of operators are operators

This follows directly from the fact that continuity and linearity are preserved under composition. What's more, however, is the following inequality.

Proposition 3.2.

$$\|ST\| \leq \|S\|\|T\|$$

Proposition 3.3. Let $T : X \to Y$

$$\|ST\| \leq \|S\|\|T\|$$

Proposition 3.4. If T is a bijective BLO then T^{-1} is a BLO, and $\|T^{-1}\| \geq \|T\|^{-1}$

3.1.1 Matrix norms

Linear maps between finite dimensional linear spaces $T : F^n \to F^m$ are examples of BLOs between normed linear spaces (one can always equip some p-norms on spaces of the form F^n, and of these the most common would be the 2-norm). This class of operators can always be represented by a matrix, where its indexes are elements of F.

Operator norms can be difficult to calculate, however for linear maps between finite dimensional linear spaces, $\|\cdot\|_{1,1}$ and $\|\cdot\|_{\infty,\infty}$ are easy to calculate and they give rise to upper and lower bounds for $\|\cdot\|_{2,2}$ giving rise to approximations of the operator norm based on matrix entries such as the *Frobenius norm*.

Definition 3.7 (Frobenius norm)**.** Let $\mathbf{A}$ be an $n \times m$ matrix, then the *Frobenius norm* is the following expression.

$$\|\mathbf{A}\|_{\text{Frob}} = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m} |[\mathbf{A}]_{ij}|^2}$$

Although it is easy to state the definition of the Frobenius norm, it is not immediately apparent in which context it is useful. As previously stated, the 2-norm is often of the most interest (especially since it is the norm resulting from the dot product on F^n, therefore being the norm of the unique Hilbert space on F^n); the Frobenius norm will eventually be found to be an upper bound of the operator norm when both spaces use a 2-norm.

$$\|\mathbf{A}\|_{1,1} = \max\{\sum_{i=1}^{n} |[\mathbf{A}_{ij}]| \in \mathbb{R} : j \in [1, m] \cap \mathbb{N}\}$$

$$\|\mathbf{A}\|_{\infty,\infty} = \max\{\sum_{j=1}^{m} |[\mathbf{A}_{ij}]| \in \mathbb{R} : i \in [1,n] \cap \mathbb{N}\}$$

$$\|\mathbf{A}\|_{2,2} \leq \|\mathbf{A}\|_{\text{Frob}}$$

- differentiation operator - differential operator - adjoint operator - bounded linear operator - operator norm

3.2 Uniform bounded principle

3.3 Spectral theorem

3.4 Hahn-Banach theorem

3.5 Open mapping theorem

3.6 Closed graph theorem

3.7 Uniform bounded principle

Chapter 4

Banach algebrae

Banach algebra; Associative algebra on a Banach space C-Algebra C-Algebra

4.1 Stone-Weierstrass theorem

The familiar Weierstrass approximation theorem is generalized considerably by the *Stone-Weierstrass theorem*; this theorem uses the theory of Banach algebrae to generalizee the function spaces used for approximating ('subalgebrae of the continuous functions that separate points' rather than space of polynomials) and applies the result for more general domain spaces (Hausdorff spaces rather than just $\mathbb{R}$).

Chapter 5

Banach and Hilbert spaces

We now focus on the deeper properties of such spaces.

5.1 Hilbert spaces

Separable Hilbert spaces have countable Schauder basis

Theorem 5.1 (Riesz representation theorem)**.** Let H be a Hilbert space, then any continuous functional $T : H \to \mathbb{R}$ is of the form $T\mathbf{x} = \langle \mathbf{x}, \mathbf{y} \rangle$

Since there is a bijective correspondence vectors and continuous functionals in Hilbert spaces (from a Hilbert space to its dual space), this gives rise to the Riesz map.

$$R : H \to H^*$$
$$R\mathbf{x} = T$$
$$T\mathbf{y} = \langle \mathbf{x}, \mathbf{y} \rangle$$

Within the finite dimension inner product spaces $\mathbb{C}^n$, it was seen that matrixes obey $\langle (\mathbf{A}^\intercal)^*\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{A}\mathbf{y} \rangle$ (where $\mathbf{A}^*$ is the conjugate of $\mathbf{A}$). One may call $(\mathbf{A}^\intercal)^*$ the *adjoint of* $\mathbf{A}$; the matrix that arises when considering a matrix on the other argument, but keeping the inner product equal.

Generalizing this notion from finite dimension inner product spaces to Hilbert spaces gives the notion of an *adjoint operator*.

Definition 5.1 (Adjoint operator)**.** The *adjoint operator of* T is the unique operator T^* satisfying the following relation.

$$\langle T^*\mathbf{y}, \mathbf{x} \rangle_X = \langle \mathbf{y}, T\mathbf{x} \rangle_Y$$

The fact that an adjoint operator of some T is unique follows from the Riesz representation theorem.

As expected, the adjoint operator shares many properties with the Hermitian matrix.

$(S+T)^* = S^* + T^*$ $T^{**} = T$ $I^* = I$ $(\lambda T)^* = \lambda^* T^*$ $(ST)^* = T^* S^*$ $\|T^*T\| = \|T\|^2$

In $\mathbb{C}^n$, the *self-adjoint matrixes* are those that satisfy $\langle \mathbf{A}\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{A}\mathbf{y} \rangle$, or equivalently $\mathbf{A} = (\mathbf{A}^\intercal)^*$. Similarly, we consider operators in Hilbert spaces that are their own adjoint operator.

Definition 5.2 (Self-adjoint operator)**.**

$$T^* = T$$

Proposition 5.1.

$$\mathbf{ker}(T^*) = \mathrm{Im}(T)^\perp$$

Proposition 5.2. Let $T : X \to X$ be an operator invariant on M, then T^* is invariant on $M^\perp$.

In the context of Hilbert spaces, one can discuss unitary isomorphisms, giving rise to unitary operators.

Definition 5.3 (Unitary operator)**.** isomorphioshm whose adjoint operator is its inverse operator.

$$U^{-1} = U^*$$

The Plancherel theorem states that the Fourier transform is unitary operator.

Here is a result that pertains not just to unitary operators, but any operator preserving the inner product.

Definition 5.4.

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle U\mathbf{x}, U\mathbf{y} \rangle \implies \|U\mathbf{x}\| = \|\mathbf{x}\|$$

5.1.1 Projections on Hilbert spaces

In Hilbert space, orthogonal projection and oblique projections exist; orthogonal projection is a Hermitian operator

Theorem 5.2 (Hilbert projection theorem)**.** Let C be a closed convex subset of the Hilbert space H, then for any $\mathbf{x}$, there is some unique $\mathbf{x}_*$ where the following holds for any $\mathbf{c} \in C$

$$\|\mathbf{x} - \mathbf{x}_*\| \leq \|\mathbf{x} - \mathbf{c}\|$$

Furthermore the map between $\mathbf{x}$ and its $\mathbf{x}_*$ is continuous

Corollary 5.1. Let C be a closed convex subset of the Hilbert space H, then for any $\mathbf{x}$, there is some unique $\mathbf{x}_*$ where the following holds for any $\mathbf{c} \in C$

Furthermore the map between $\mathbf{x}$ and its $\mathbf{x}_*$ is continuous

5.2 Banach spaces

Part II

Advanced

Chapter 6

W^*-algebrae

Chapter 7

Spectral theory

Chapter 8

More function spaces

8.1 H^p space

8.2 Fréchet space (functional analysis)